COMP6834: Advanced topics in optimization
Tutorial 1: the simplex algorithm

1 First example

Let us solve the following simple example:

maximize $z = 2x_1 + 3x_2$
subject to:

$$\begin{array}{lll} x_1 & +x_2 & \leq 6 \\ 2x_1 & +x_2 & \leq 10 \\ -x_1 & +x_2 & \leq 4 \end{array}$$

$$x_1, x_2 \geq 0$$

In order to put it into the standard form (equations instead of inequalities), we add three *slack variables* x_3, x_4, x_5, and obtain the following equivalent system:

maximize $z = 2x_1 + 3x_2$
subject to:

$$\begin{array}{llllll} x_1 & +x_2 & +x_3 & & & = 6 \\ 2x_1 & +x_2 & & +x_4 & & = 10 \\ -x_1 & +x_2 & & & +x_5 & = 4 \end{array}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

In order to start the simplex algorithm, we need an easy feasible solution. Basically, if we are able to find in each constraint a variable which appears in this constraint only, then you can easily find a feasible solution.
In our case: $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 6, 10, 4)$ is a feasible solution. Variables having non-zero values are called the *basics* variables. We construct the following tableau:

	basis	z	x_1	x_2	x_3	x_4	x_5	RHS
L_1	z	1	-2	-3	0	0	0	0
L_2	x_3	0	1	1	1	0	0	6
L_3	x_4	0	2	1	0	1	0	10
L_4	x_5	0	-1	1	0	0	1	4

Concerning the first row, we write the objective function in a constraint-like way: $z - 2x_1 - 3x_2 = 0$. RHS stands for "Right Hand Side" of the equations. In the first column, we write the basic variable corresponding to the constraint.
As we can see, the z column together with the columns of the basis form the identity matrix as a submatrix. When this happens, we say that the tableau is in a *legitimate* form.
The simplex algorithm consists in several iterations, each of them being summarized as follows:

- replace a basic variable by a non-basic variable
- modify the tableau in order to make it legitimate

Concerning the first item, we need to find the non-basic variable which will increase the most the objective function. Thus, we look at the z row for the column **with the most negative value** (for a minimization problem, we would look for the column with the least negative value). This column corresponds to the variable which will enter the basis. In the example above, x_2 is the entering variable (that is why the corresponding column has been colored in red).
In order to find the leaving variable, we look for the first constraint which becomes tight when increasing the value of the entering variable. That is, for each row, we divide the right hand side value by the value in the column corresponding to the entering variable, and we look for **the minimum positive value over all ratios** (we do not compute the ratio for the z row). We have the following:

	basis	z	x_1	x_2	x_3	x_4	x_5	RHS	Min.
L_1	z	1	-2	-3	0	0	0	0	ratio
L_2	x_3	0	1	1	1	0	0	6	6/1
L_3	x_4	0	2	1	0	1	0	10	10/1
L_4	x_5	0	-1	1	0	0	1	4	4/1

In the example above, x_5 is the leaving variable. The value at the intersection (in green) is called the *pivot*.
Concerning the second item, we need to modify the tableau in order to make it legitimate. That is, such that the column corresponding to the entering variable has a 1 at the pivot and a 0 elsewhere. To do so, we use the row corresponding to the leaving variable. Concerning the row L_4, we do not need to do anything, since the pivot value is already 1. Then we proceed to the following operations:

- $L_1 \leftarrow L_1 + 3L_4$
- $L_2 \leftarrow L_2 - L_4$
- $L_3 \leftarrow L_3 - L_4$

And we obtain the following tableau:

	basis	z	x_1	x_2	x_3	x_4	x_5	RHS	Min.
L_1	z	1	-5	0	0	0	3	12	ratio
L_2	x_3	0	2	0	1	0	-1	2	2/2
L_3	x_4	0	3	0	0	1	-1	6	6/3
L_4	x_2	0	-1	1	0	0	1	4	(neg.)

Now, x_1 is the entering variable, x_3 is the leaving variable. We need to divide L_2 by 2, and make the following operations (notice that these changes are made after the modification of L_2):

- $L_1 \leftarrow L_1 + 5L_2$
- $L_3 \leftarrow L_3 - 3L_2$
- $L_4 \leftarrow L_4 + L_2$

And obtain the following tableau:

	basis	z	x_1	x_2	x_3	x_4	x_5	RHS
L_1	z	1	0	0	5/2	0	1/2	17
L_2	x_1	0	1	0	1/2	0	-1/2	1
L_3	x_4	0	0	0	-1/2	1	1/2	3
L_4	x_2	0	0	1	1/2	0	1/2	5

Here, all values of the z row are positive: the algorithm stops, and a solution has been found: its value is 17 (right hand side value of the z row), and the solution is $(x_1, x_2, x_3, x_4, x_5) = (1, 5, 0, 3, 0)$ (the value of the basic variables are in the RHS column, the value of non-basic variables are 0). In order to obtain a solution to the first linear program, we just take the projection to (x_1, x_2). Thus, we just remove the slack variables, and obtain the solution $(x_1, x_2) = (1, 5)$.

2 Second example

We would like to solve the following linear program:

maximize $z = 1000x_1 + 1200x_2$
subject to:

$$\begin{array}{lll} 10x_1 & +5x_2 & \leq 200 \\ 2x_1 & +3x_2 & = 60 \\ x_1 & & \leq 12 \\ & x_2 & \geq 6 \end{array}$$

$$x_1, x_2 \geq 0$$

We turn it into the standard form by adding three slack variables:

maximize $z = 1000x_1 + 1200x_2$
subject to:

$$\begin{array}{llllll} 10x_1 & +5x_2 & +e_1 & & & = 200 \\ 2x_1 & +3x_2 & & & & = 60 \\ x_1 & & & +e_3 & & = 12 \\ & x_2 & & & -e_4 & = 6 \end{array}$$

$$x_1, x_2, e_1, e_3, e_4 \geq 0$$

However here, we cannot find any easy feasible solution. Thus, before finding an optimal solution to this problem, the first step consists in finding such a feasible solution.

Phase I: To do so, we add two *artificial* variables and change the objective function:

maximize ~~$z = 1000x_1 + 1200x_2$~~ $z_A = -a_2 - a_4$
subject to:

$$\begin{array}{llllllll} 10x_1 & +5x_2 & +e_1 & & & & & = 200 \\ 2x_1 & +3x_2 & & & & +a_2 & & = 60 \\ x_1 & & & +e_3 & & & & = 12 \\ & x_2 & & & -e_4 & & +a_4 & = 6 \end{array}$$

$$x_1, x_2, e_1, e_3, e_4, a_2, a_4 \geq 0$$

The previous linear program has a feasible solution if and only if this new program has optimal value 0 (i.e. when $a_2 = a_4 = 0$).
Here we have the feasible solution $(x_1, x_2, e_1, e_3, e_4, a_2, a_4) = (0, 0, 200, 12, 0, 60, 6)$ and the basics variables are thus $\{e_1, a_2, e_3, a_4\}$. We have the following tableau (recall that for the first row z_A, we write the objective function in a constraint-like way: $z_A + a_2 + a_4 = 0$):

	basis	z_A	x_1	x_2	e_1	e_3	e_4	a_2	a_4	RHS
L_1	z_A	1	0	0	0	0	0	1	1	0
L_2	e_1	0	10	5	1	0	0	0	0	200
L_3	a_2	0	2	3	0	0	0	1	0	60
L_4	e_3	0	1	0	0	1	0	0	0	12
L_5	a_4	0	0	1	0	0	-1	0	1	6

However it is not in a legitimate form: the z column together with the columns corresponding to basic variables do not form the identity matrix (because of the z_A row mainly). We thus modify the first row $L_1 \leftarrow L_1 + L_3 + L_5$ (notice that most of the time, we just have to add to L_1 the rows in which we added artificial variables, L_3 and L_5 here). We obtain the following tableau and are now ready to start the iterations:

	basis	z_A	x_1	x_2	e_1	e_3	e_4	a_2	a_4	RHS	min.
L_1	z_A	1	-2	-4	0	0	1	0	0	-66	ratio
L_2	e_1	0	10	5	1	0	0	0	0	200	200/5
L_3	a_2	0	2	3	0	0	0	1	0	60	60/3
L_4	e_3	0	1	0	0	1	0	0	0	12	-
L_5	a_4	0	0	1	0	0	-1	0	1	6	6/1

Here, x_2 is the entering variable, and a_4 is the leaving variable. We then turn it into a legitimate form: L_5 does not change, and:

- $L_1 \leftarrow L_1 + 4L_5$

- $L_2 \leftarrow L_2 - 5L_5$
- $L_3 \leftarrow L_3 - 3L_5$
- $L_4 \leftarrow L_4$

	basis	z_A	x_1	x_2	e_1	e_3	e_4	a_2	a_4	RHS	min.
L_1	z_A	1	-2	0	0	0	-3	0	4	-42	ratio
L_2	e_1	0	10	0	1	0	5	0	-5	170	170/5
L_3	a_2	0	2	0	0	0	3	1	-3	42	42/3
L_4	e_3	0	1	0	0	1	0	0	0	12	-
L_5	x_2	0	0	1	0	0	-1	0	1	6	-

Here, e_4 is the entering variable, and a_2 is the leaving variable. The modifications: we divide L_3 by 3, and, using this new L_3, we get:

- $L_1 \leftarrow L_1 + 3L_3$
- $L_2 \leftarrow L_2 - 5L_3$
- $L_4 \leftarrow L_4$
- $L_5 \leftarrow L_5 + L_3$

And we obtain:

	basis	z_A	x_1	x_2	e_1	e_3	e_4	a_2	a_4	RHS
L_1	z_A	1	0	0	0	0	0	1	1	0
L_2	e_1	0	20/3	0	1	0	0	-5/3	-10	100
L_3	e_4	0	2/3	0	0	0	1	1/3	-1	14
L_4	e_3	0	1	0	0	1	0	0	0	12
L_5	x_2	0	2/3	1	0	0	0	1/3	0	20

All values in the z_A row are positive, so the algorithms stops and an optimal solution has been found. Good news: its value is 0, which was the objective. This implies that the first linear program has a feasible solution: $(x_1, x_2, e_1, e_3, e_4, a_2, a_4) = (0, 20, 100, 12, 14, 0, 0)$.

Phase II: We now remove the artificial variables, and solve the linear program with the former objective function $z - 1000x_1 - 1200x_2 = 0$:

	basic	z_A	x_1	x_2	e_1	e_3	e_4	RHS
L_1	z_A	1	-1000	-1200	0	0	0	0
L_2	e_1	0	20/3	0	1	0	0	100
L_3	e_4	0	2/3	0	0	0	1	14
L_4	e_3	0	1	0	0	1	0	12
L_5	x_2	0	2/3	1	0	0	0	20

We first turn it into legitimate form (the x_2 column should have a 0 at the first row), by changing $L_1 \leftarrow L_1 + 1200L_5$

	basis	z_A	x_1	x_2	e_1	e_3	e_4	RHS	min.
L_1	z_A	1	-200	0	0	0	0	24000	ratio
L_2	e_1	0	20/3	0	1	0	0	100	100 $\times$ 3/20
L_3	e_4	0	2/3	0	0	0	1	14	14 $\times$ 3/2
L_4	e_3	0	1	0	0	1	0	12	12
L_5	x_2	0	2/3	1	0	0	0	20	20 $\times$ 3/2

Now, x_1 is the entering variable, and e_3 is the leaving variable. We do not change L_4, and proceed to the following changes:

- $L_1 \leftarrow L_1 + 200L_4$
- $L_2 \leftarrow L_2 - 20/3 \times L_4$
- $L_3 \leftarrow L_3 - 2/3 \times L_4$
- $L_5 \leftarrow L_5 - 2/3 \times L_4$

And we obtain the tableau:

	basis	z_A	x_1	x_2	e_1	e_3	e_4	RHS
L_1	z_A	1	0	0	0	200	0	26400
L_2	e_1	0	0	0	0	-20/3	0	20
L_3	e_4	0	0	0	0	-2/3	1	6
L_4	x_1	0	1	0	0	1	0	12
L_5	x_2	0	0	1	0	-2/3	0	12

The z row has no negative value: the algorithm stops and an optimal solution has been found, of cost 26400. The solution is $(x_1, x_2) = (12, 12)$.