Laboratoire de mécanique des fluides et d'acoustique
Fluides Complexes et Transferts

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Numerical schemes for Low Mach Number flows

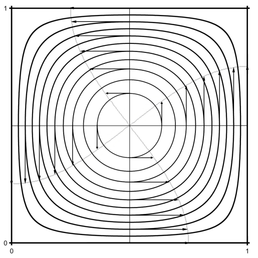
Motivation

Accuracy and convergence of low Mach number flow simulations

Low Mach number flows are difficult to compute accurately with numerical methods developed for high speed flows. Since the flow time scales may be very different at low Mach number, explicit methods require the use of very small time steps. On the other hand, implicit methods can be cumbersome because of the ill-conditioned system to solve. Preconditioning techniques exist, but modify the properties of the numerical schemes and the pertinence of those approaches for long time integration remains a particular issue.

The objective of this study is to analyse the low Mach number behaviour of a selection of numerical schemes The inspected methods use explicit or implicit time integration, with a Finite Volume, Finite Element or Finite Difference spatial discretization and are either centered or upwind schemes derived from Godunov's method. The study focuses on the application to the simulation of the 2D compressible Taylor vortex flow, solution of the compressible Euler (inviscid) equations.

Initial conditions



A simple flow : the inviscid, 2D, Taylor vortex

$$\begin{aligned} \mathbf{u}(\mathbf{x}, t=0) &= \mathbf{u}_0(\mathbf{x}) = (\partial_y\psi_0, -\partial_x\psi_0) \\ P(\mathbf{x}, t=0) &= 1 + M^2 P_2(\mathbf{x}) \\ \rho(\mathbf{x}, t=0) &= 1 \\ \text{with} \quad \psi_0 &= \pi^{-1}\sin(\pi x)\sin(\pi y) \\ \text{and} \quad \nabla P_2 &= -\mathbf{u}_0.\nabla\mathbf{u}_0 \end{aligned}$$

Streamlines at $t = 0$ ($\psi_0 = C^t$).
$\mathbf{u}.\mathbf{n} = 0$ at the boundary.

Asymptotic expansion

Low Mach number M expansion using 2 time-scales (t, $\tau = t/M$)

$$\begin{aligned} \mathbf{u}(\mathbf{x}, t) &= \mathbf{u}_0(\mathbf{x}) + M^2\left[\overline{\mathbf{u}}_2(\mathbf{x}, t) + \delta\mathbf{u}_2(\mathbf{x}, \tau)\right] + M^3\delta\mathbf{u}_3(\mathbf{x}, \tau, t) + \cdots \\ P(\mathbf{x}, t) &= 1 + M^2 P_2(\mathbf{x}) + M^3\boxed{\delta P_3(\mathbf{x}, \tau)} + M^4\left[\boxed{\overline{P}_4(\mathbf{x}, t)} + \delta P_4(\mathbf{x}, \tau, t)\right] + \cdots \\ \rho(\mathbf{x}, t) &= 1 + M^2\boxed{\overline{\rho}_2(\mathbf{x}, t)} + M^3\delta\rho_3(\mathbf{x}, \tau) + M^4\left[\overline{\rho}_4(\mathbf{x}, t) + \delta\rho_4(\mathbf{x}, \tau, t)\right] + \cdots \end{aligned}$$

entropic terms, acoustic terms, $\overline{(.)}$ denotes average over the τ time scale.

At $t = 0$, entropy has $O(M^2)$ variations throughout the flow. At the leading order, density variations result from entropy advection along the closed streamlines of the $\mathbf{u}_0$ field. Density variations at $O(M^2)$ have the time-period $O(1)$. Due to stretching by the flow, these variations are such that their radial length scales decrease as t^{-1} and displays a striped pattern. Since material derivative of density is not zero during this process, a compressible contribution is generated in the velocity field on the time scale $O(1)$. The flow adapts to the divergence-free conditions at $t = 0$ by generating an acoustic response on the short time scale $\tau = t/M$ characterized by $O(M^2)$ variations for the velocity field and $O(M^3)$ for density and pressure.

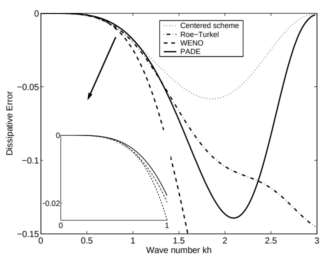
Numerical schemes

Four different schemes representative of numerical methods for compressible flows are evaluated using three spatial discretization techniques :

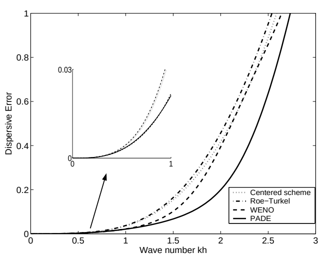
(a) 2nd order implicit centered scheme with FV approximation,
(b) 2nd order explicit (upwind) Roe-Turkel scheme with FE/FV approximation,
(c) 4th-order explicit (centered) Padé scheme with FD approximation,
(d) 5th-order explicit (upwind) WENO scheme with FD approximation.

Spatial and temporal discretization errors are evaluated in the case of one dimensional linear advection on a regular mesh of size h. Dissipation and dispersion errors are evaluated using standard Fourier analysis for a propagating wave of wavenumber k.

- Centered schemes (implicit centered scheme and compact Padé scheme) are less dissipative than the two upwind schemes.
- High-order schemes (Padé and WENO) have less dispersion error than second-order schemes.
- FV 2nd order schemes are relatively accurate as compared to the high order FD schemes and especially much more precise than the standard 2nd order FD scheme.



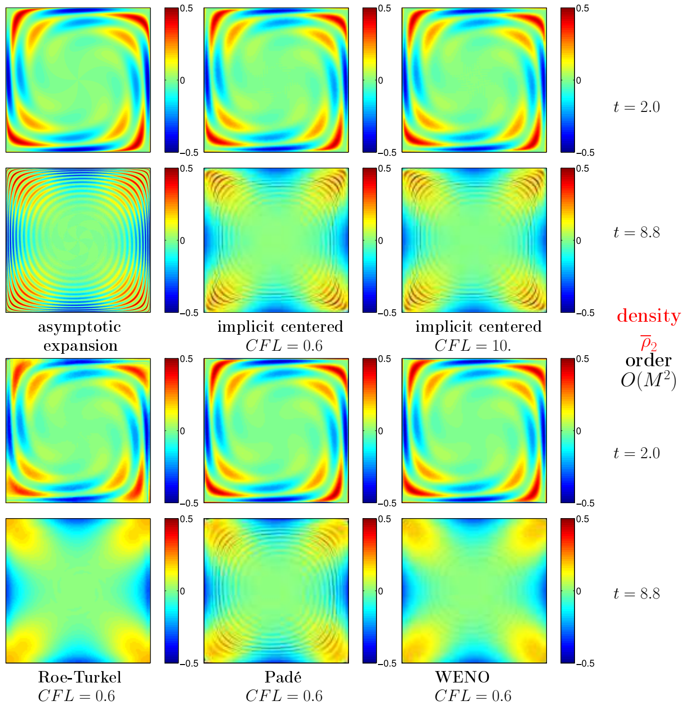
Dissipative error



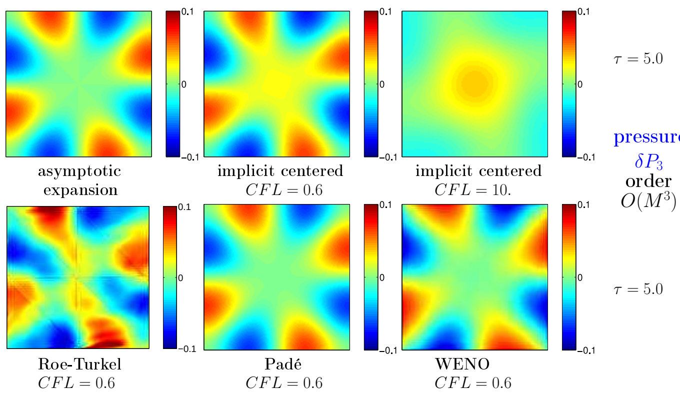
Dispersive error

Spectral error analysis for the 1D linear advection equation for CFL = 0.6

Results



$\overline{\rho}_2$ term on a 120×120 grid at $M = 0.01$.



δP_3 term on 120×120 grid at $M = 0.01$

Comments

Comparison of schemes

- Explicit 2nd scheme, even with low Mach number preconditioning: the most dissipative but robust
- Implicit 2nd centered scheme: captures the slow-time density variations without dissipation and with a small dispersion error. For CFL smaller than one, acoustic pressure levels are correctly predicted. For CFL larger than one, acoustic fluctuations are filtered without alteration of the slow-time solution.
- Explicit high order schemes: both correctly predict the slow time density variations on a rather coarse mesh size. Even if the high-order Padé centered scheme needs low pass spatial filtering, it is less dissipative than the WENO-Roe upwind scheme and provides the best results.
- Implicit high order finite volume centered schemes seems a promising way to combine the advantages of both approaches.

Reference

Asymptotic and numerical analysis of an inviscid bounded vortex flow at low Mach.
A. Cadiou, L. Le Penven, M. Buffat (2008) *J. Comput. Phys.* vol. 227, pp8268-8289.