Comportement en temps long d'équations cinétiques avec effets de bord

Thèse encadrée par Nicolas Fournier (SU-LPSM) et Stéphane Mischler (PSL-CEREMADE)

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Long-Time Behavior of Kinetic Equations with Boundary Effects

Kinetic Theory
- Introduction
- Free molecular flow
- Boundary conditions

Probabilistic approach for the asymptotic behavior of the FTE

Analytic method, Harris' theorems

Linearized equations from collisional kinetic theory

Outlook

Kinetic theory I

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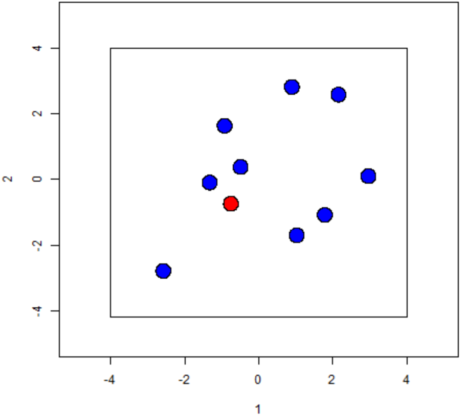
One possible model: a system of hard-spheres with no outside force and a simple boundary condition.

Kinetic theory II



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2. Studying microscopically the system makes it difficult to understand the behavior of macroscopic quantities, e.g. the temperature.

In kinetic theory, we adopt a statistical view of the system. The idea is to analyze the behavior of a "typical" particle, rather than trying to follow all of them. This is also the good point of view for the study of the convergence towards some equilibrium.

A basic example: free molecular flow

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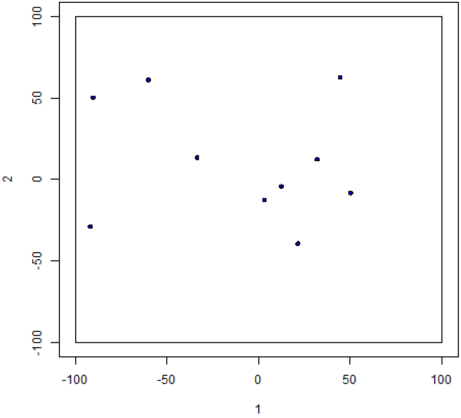
This is the (kinetic) free-transport equation inside the bounded domain $\Omega \subset \mathbb{R}^d$. It must be completed with

- an initial condition $f_0(\cdot,\cdot)$ on $\Omega \times \mathbb{R}^d$;
- some conditions at the boundary $\partial\Omega$ of the spatial domain.

A view of the free-molecular flow



Boundary conditions

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Specular reflection

Let $x \in \partial\Omega$ the boundary of Ω, n_x the unit outward normal vector at x, $v \in \mathbb{R}^d$ such that $v \cdot n_x > 0$, then

$$\eta_x(v) = v - 2(v \cdot n_x)n_x,$$

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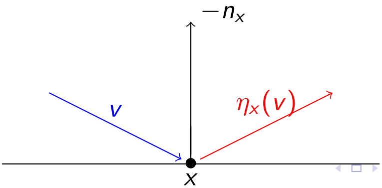

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For instance, with such condition, the gas exerts no stress on the boundary in the tangential directions. In practice, a surface tension is observed.
To understand why this model is not accurate, one needs to remember that the wall is itself made of molecules ! And possibly, of several layers of spaced ones...

Diffuse reflection

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Diffuse reflection

Let $\Sigma := \partial\Omega \times \mathbb{R}^d$. At the boundary, the density f satisfies, for $(t, x, v) \in \mathbb{R}_+ \times \Sigma$, with $v \cdot n_x < 0$,

$$f(t, x, v) = cM(v)\widetilde{\gamma_+ f}(t, x),$$

where c is a normalizing constant and $\widetilde{\gamma_+ f}$ is the flux, given by

$$\widetilde{\gamma_+ f}(t, x) = \int_{\{v' \cdot n_x > 0\}} f(t, x, v')|v' \cdot n_x| \mathrm{d}v'.$$

The kernel M

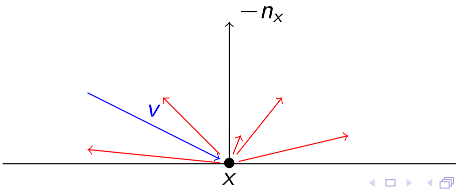
The most interesting case is the wall Maxwellian:

$$M(v) = \frac{e^{-\frac{|v|^2}{2}}}{(2\pi)^{\frac{d}{2}}}.$$

Extensions/other choices are possible:

1. a dependency of the temperature in x. Then $c(x)M(x,v) = c(x)e^{-\frac{|v|^2}{2\theta(x)}}$ with $\theta(x)$ the temperature at $x \in \partial\Omega$, $c(x)$ a normalizing constant.
2. Stochastic billards: M conserves energy, but no radial symmetry.

Hereafter we assume radial symmetry and continuity around 0.



The Maxwell boundary condition

Combining both conditions gives a more accurate description.

Maxwell boundary condition

For $(t, x, v) \in \mathbb{R}_+ \times \Sigma$ with $v \cdot n_x < 0$,

$$f(t, x, v) = (1 - \alpha(x)) f(t, x, \eta_x(v)) + \alpha(x) c M(v) \widetilde{\gamma_+ f}(t, x),$$

with $\alpha(x)$ the **accommodation** coefficient at $x \in \partial\Omega$.

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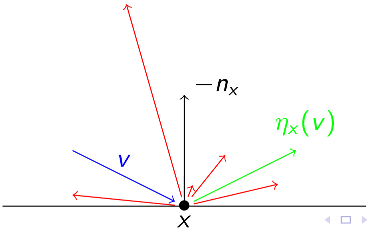

Long-Time Behavior of Kinetic Equations with Boundary Effects

Model and hypotheses

We consider the free-transport equation with Maxwell boundary condition, constant temperature, $d \geq 2$. We assume that Ω has volume 1 in what follows. We set $G := \Omega \times \mathbb{R}^d$ and let $f_0 \in L^1(G)$.

It is known (Arkeryd-Cercignani) that the equation admits a unique solution f such that $f(t, \cdot, \cdot) \in L^1(G)$ for all $t \geq 0$. Alternatively we may work with measures, but there is then no uniqueness. We want to understand the behavior, when $t \to \infty$, of this solution.

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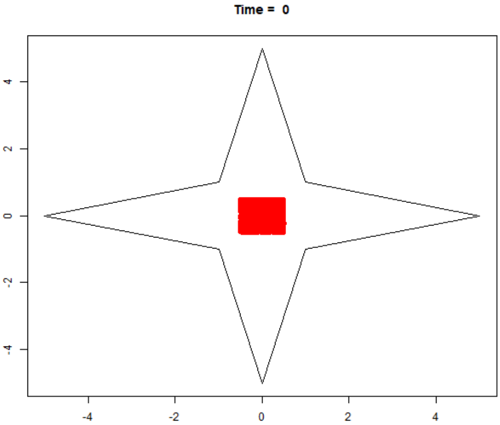
A first key hypothesis: for all $x \in \partial\Omega$, $\alpha(x) \geq \alpha_0$ for some $\alpha_0 > 0$. If $\alpha \equiv 0$, no equilibrium.

The purely specular case



Qualitative convergence towards equilibrium

The system has a natural entropy: setting $W(t) = \int_G f \ln(\frac{f}{M}) \, \mathrm{d}v\mathrm{d}x \geq 0$,

$$\frac{d}{dt} W(t) \leq 0.$$

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Also, if $f(t,x,v) = M(v)$ for all $(t,x,v) \in \mathbb{R}_+ \times G$, $W(t) = 0$ for all $t \geq 0$.
One can in fact show (Arkeryd-Nouri) that, starting with f_0 having mass 1, regular enough, f converges towards

$$f_\infty(x,v) = M(v).$$

Key question: what is the rate at which this convergence occurs in the L^1 norm? Slow velocities persist a long time $\rightarrow$ no exponential convergence.

Known results for this problem (wall Maxwellian)

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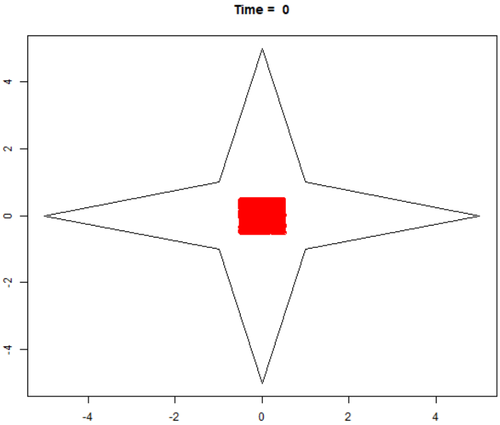
A convergence in a non-symmetric domain



First strategy: the coupling method [Chapter 2]

Key idea 1: it is possible to build a process $(X_t, V_t)_{t\geq 0}$ whose law is a solution to the free-transport equation with Maxwell boundary condition. Some randomness appears in this construction.

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Key idea 2 (technically harder): we can find a coupling (i.e. two constructions with correlated randomnesses) $(X_t, V_t, \tilde{X}_t, \tilde{V}_t)_{t\geq 0}$ s.t. $(X_0, V_0) \sim f_0$, $(\tilde{X}_0, \tilde{V}_0) \sim f_\infty$ and, for

$$\tau = \inf\{t > 0, (X_{t+s}, V_{t+s})_{s\geq 0} = (\tilde{X}_{t+s}, \tilde{V}_{t+s})_{s\geq 0}\},$$

we can show (when M is the Maxwellian wall) the inequality $\mathbb{E}[\tau^{d-}] < \infty$.

Conclusion from the coupling

Once the control on τ is established, we conclude using properties of the total variation distance: if $(X_t, V_t) \sim f_t$ the solution at time t and $(\tilde{X}_t, \tilde{V}_t) \sim f_\infty$, then

$$\begin{aligned}\|f_t - f_\infty\|_{TV} &= \inf_{(X,V)\sim f_t, (\tilde{X},\tilde{V})\sim f_\infty} \mathbb{P}((X,V) \neq (\tilde{X},\tilde{V})) \\ &\leq \mathbb{P}(\tau > t) \leq \frac{\mathbb{E}[(\tau+1)^{d-}]}{(t+1)^{d-}},\end{aligned}$$

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This strategy also allows one to work in the framework of measures, although the solution f_t is not unique in this case.

Results

Theorem (B., Fournier)

Let Ω be a C^2 bounded domain, $G := \Omega \times \mathbb{R}^d$. Let $f_0 \in L^1(G)$ and write f_t for the unique solution at time $t \geq 0$. If $r : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing with $r(x+y) \lesssim r(x) + r(y)$, and

$$\int_G r\Big(\frac{1}{|v|}\Big) f_0(x,v)\,\mathrm{d}v\mathrm{d}x + \int_G r\Big(\frac{1}{|v|}\Big) M(v)\,\mathrm{d}v\mathrm{d}x < \infty,$$

then, for all $t \geq 0$, for some constant $C > 0$,

$$\|f_t - f_\infty\|_{L^1} \leq \frac{C}{r(t)}.$$

Numerical study of the asymptotic behavior [Chapter 3]

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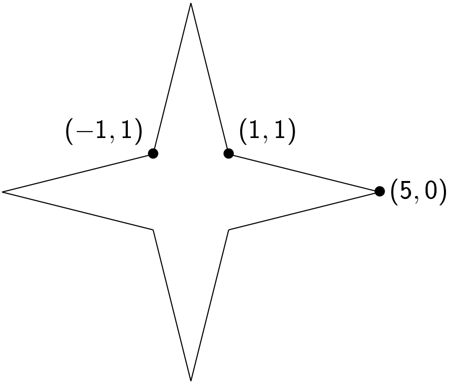
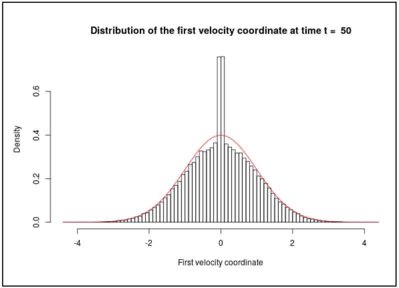
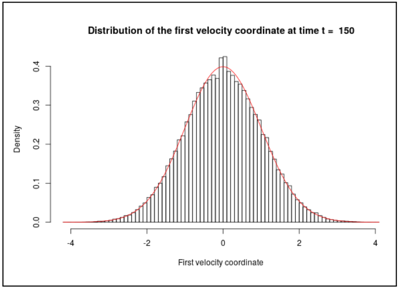
We will focus on the following star-shaped domain (2D).



Qualitative behavior

Initial distribution hereafter: uniform distribution in space, law $\mathcal{N}(0, 0.01 I_2)$ for the velocity
Parameters: 10^6 particles, $\alpha \equiv 1$ (pure diffuse reflection), M the Maxwellian wall.





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We will rather use the following property: if μ, ν are two measures on a measurable space $(E, \mathcal{E})$,

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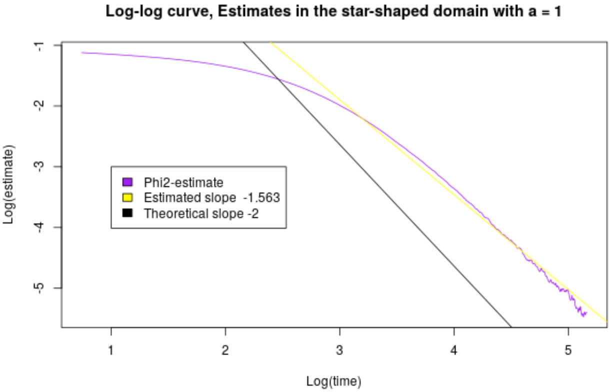
Hence we can approximate the total variation by testing the distribution against a function ϕ. In what follows we present estimates corresponding to the choice $\phi_2(x, v) = \sqrt{|x|} + \sqrt{|v|}$.

Case of the wall Maxwellian



This is a log-log curve -> we have a polynomial rate, as expected.
However, its value is far from the theoretical prediction.

Changing the distribution at the boundary

Instead of using the wall Maxwellian

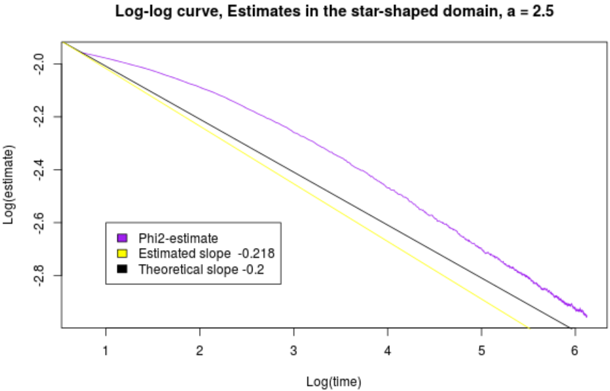
$$M(v) = \frac{e^{-\frac{|v|^2}{2}}}{2\pi}, \qquad v \in \mathbb{R}^2,$$

we can modify slightly the distribution to obtain more or less concentration around 0:

$$M_a(v) \propto e^{-\frac{|v|^{\frac{2}{a}}}{2}} |v|^{\frac{3}{a}-3}, \qquad v \in \mathbb{R}^2, \quad a \in (0,3).$$

This changes the rate of convergence if the initial data is also adapted. In particular, with the previous initial data, we expect an exponent of the rate equal to $\frac{3}{a} - 1$ for $a \in]1,3[$ (the problem is slightly more complicated for $a < 1$).

An example: the case $a = 2.5$.



We clearly see the difference with respect to the case $a = 1$! Once again, the empirical rate differs slightly from the theoretical one.

Long-Time Behavior of Kinetic Equations with Boundary Effects

Kinetic Theory

Probabilistic approach for the asymptotic behavior of the FTE

Analytic method, Harris' theorems

- Two sub-geometric Harris' theorems
- Back to the free-transport equation [Chapter 4]
- Linking coupling and Lyapunov criteria [Chapter 5]

Linearized equations from collisional kinetic theory

Outlook

Same problem, different strategy: Harris' sub-geometric theorem

Another way to study the convergence of the free-transport equation is to apply Harris' theorem, more precisely the deterministic sub-geometric version of Cañizo and Mischler (following the probabilistic results of Douc-Fort-Guillin and Hairer-Mattingly).

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Another way to study the convergence of the free-transport equation is to apply Harris' theorem, more precisely the deterministic sub-geometric version of Cañizo and Mischler (following the probabilistic results of Douc-Fort-Guillin and Hairer-Mattingly).
In what follows, M is the wall Maxwellian, but the temperature is allowed to vary at the boundary. We assume again, for all $x \in \partial\Omega$, $\alpha(x) \geq \alpha_0$ for some $\alpha_0 > 0$.

Sub-geometric Harris' theorem from the probabilistic side

Let $(X_t)_{t\geq 0}$ be a Borel right process with values in $(E, \mathcal{E})$, with associated Markov semigroup $(\mathcal{P}_t)_{t\geq 0}$, generator $\mathcal{L}$, non-explosive, irreducible, aperiodic. Then if

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1. $\exists C \in \mathcal{E}$ compact and petite, i.e. there $\exists a \in \mathcal{P}(\mathbb{R}_+)$ and a σ-finite measure $\nu \not\equiv 0$ on $\mathcal{E}$ such that

$$\forall x \in C, \int_0^\infty \mathcal{P}_t(x, \cdot) a(dt) \geq \nu(\cdot);$$

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$$\forall x \in C, \int_0^\infty \mathcal{P}_t(x,\cdot)a(dt) \geq \nu(\cdot);$$

2. $\mathcal{L}V \leq -\phi(V) + K1_C$, for some $K \geq 0$, for $\phi \uparrow \infty$, strictly concave (+ technical requirements)

letting $H_\phi(u) = \int_1^u \frac{du}{\phi(u)}$, $\exists \pi$ invariant measure for $(\mathcal{P}_t)_{t\geq 0}$ on E and $C > 0$ s.t. for all $x \in E$,

$$\lim_{t\to\infty} \phi(H_\phi^{-1}(t))\|\mathcal{P}_t(x,\cdot) - \pi(\cdot)\|_{TV} = 0.$$

A deterministic result

Deterministic sub-geometric Harris Theorem (Cañizo-Mischler)

With the same notations, if, $1 \leq m_0 \lesssim m_1 \lesssim m_2 \lesssim m_3$ are four weights with

$$\mathcal{L}^* m_1 \leq -m_0 + K_0, \qquad \mathcal{L}^* m_3 \leq -m_2 + K_2, \qquad K_0, K_2 > 0,$$

and if, for any $R > R_0 > 0$, there exist $T \geq T_0$ and a measure $\nu \not\equiv 0$ such that

$$e^{\mathcal{L}^* T} f \geq \nu \int_{\{|x| \leq R\}} f \,\mathrm{d}x, \qquad \forall f \in L^1(E)_+ = \{f \in L^1(E), f \geq 0\},$$

then a quantitative rate of convergence, based on an interpolation condition holding between the weights $(m_i)_{i \geq 1}$ can be obtained.

Application to the free-transport problem

E.g., if we let $\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}$ and if $0 < \delta \leq k$ with $m_0 \simeq 1$, $m_1 \simeq \langle x \rangle^{\delta}$, $m_2 \simeq \langle x \rangle^{k-\delta}$ and $m_3 \simeq \langle x \rangle^{k}$ we can show that the final rate is $t^{-\frac{k}{\delta}}$.

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For the free-transport equation, the key quantity to look at is

$$\sigma(x, v) = \inf\{t > 0, x + tv \in \partial\Omega\},$$

which is the time it takes, for a particle starting at 0 in position x with velocity v, to hit the boundary.

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$$\sigma(x,v) = \inf\{t > 0, x + tv \in \partial\Omega\},$$

which is the time it takes, for a particle starting at 0 in position x with velocity v, to hit the boundary.

An important fact is that $v \cdot \nabla_x \sigma(x,v) = -1$. This will be the main ingredient of the Lyapunov inequalities. Let $\langle x, v \rangle = (1 + \sigma(x,v))$ on $\bar{G}$ in what follows.

Application to the free-transport problem II

Idea: let $m_3(x,v) = \langle x, v\rangle^k$ in $\bar{G}$, $m_2(x,v) = k\langle x, v\rangle^{k-1}$. Let $\|\cdot\|_m$ be the weighted L^1 norm with weight $m \geq 1$. Then, for $T > 0$, $\int_G f_0 = 0$,

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\begin{aligned}
\frac{d}{dt}\|f_T\|_{m_3}\,\mathrm{d}v\mathrm{d}x &\leq -\int_G (v\cdot\nabla_x)|f_T|m_3\,\mathrm{d}v\mathrm{d}x \\
&= \int_G |f_T|(v\cdot\nabla_x)m_3\,\mathrm{d}v\mathrm{d}x - \int_\Sigma (v\cdot n_x)|f_T|m_3\,\mathrm{d}v\mathrm{d}\zeta(x) \\
&\leq -\|f_T\|_{m_2} + \int_{\partial\Omega}\widetilde{\gamma_+ f}\underbrace{\int_{\{v'\cdot n_x<0\}} M(v')|v'\cdot n_x|\Big(1+\frac{\mathrm{diam}(\Omega)}{|v'|}\Big)^k\mathrm{d}v'}_{=C_k}\,\mathrm{d}\zeta(x).
\end{aligned}
$$

We see that $\delta = 1$ with the previous notations. Two issues:

1. Controlling the flux part $\to$ integrated inequalities.
2. The right quantity to look at is not a norm but rather $1+\sigma(x,v)$ which behaves as $\frac{1}{|v|}$ for small velocities.

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Note that $f_\infty \equiv M \in L^1_{\langle x,v\rangle^{d-}}(G) \setminus L^1_{\langle x,v\rangle^{(d+1)-}}(G)$ so take $k = d-$

$\rightarrow$ again, rate in $t^{-\frac{k}{\delta}} = t^{-(d-)}$.

A theorem in the exponential case (Meyn-Tweedie, ...)

Theorem

With $(X_t)_{t\geq 0}$ *as in Slide 30, the following conditions are equivalent:*
1. there exist a compact petite set $C \in \mathcal{E}$ *and* $\delta > 0$, $\kappa > 1$ *such that, for* $\tau_C(\delta) = \inf\{t > \delta : X_t \in C\}$,

$$\forall x \in E, \mathbb{E}_x[\kappa^{\tau_C(\delta)}] < \infty, \qquad \textit{and} \qquad \sup_{x\in C} \mathbb{E}_x[\kappa^{\tau_C(\delta)}] < \infty;$$

2. there exist a compact petite set $C \in \mathcal{E}$, $b, \beta > 0$ *and* $V : E \to [1, \infty]$, *finite at some* $x_0 \in E$ *such that*

$$\mathcal{L}V(x) \leq -\beta V(x) + b1_C(x), \qquad \forall x \in E.$$

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Each condition implies that there exist $\rho < 1$, $d > 0$ and an invariant measure π on E such that, for all $x \in E$,

$$\|\mathcal{P}_t(x, \cdot) - \pi(\cdot)\|_{TV} \leq dV(x)\rho^t.$$

The sub-geometric version (Douc-Fort-Guillin, ...)

Theorem

With $(X_t)_{t\geq 0}$ *as in Slide 30, let* $\phi : [1,\infty) \to (0,\infty)$ *increasing, differentiable, concave. Let* H_ϕ *defined as before. Consider the two conditions*

1. $\exists \delta > 0$ *and a compact petite set* C *such that for all* $x \in E$, $\mathbb{E}_x[\int_0^{\tau_C(\delta)} H_\phi^{-1}(s)\mathrm{d}s] < \infty$, *with a uniform bound on* C*;*

2. there exist a compact petite set C*, a constant* $b < \infty$ *and* $V : X \to [1,\infty)$ *unif. bounded on* C *such that*

$$\mathcal{L}V \leq -\phi(V) + b1_C.$$

Both conditions imply that there exists an invariant measure π *on* E *s.t. for all* $x \in E$,

$$\lim_{t\to\infty} \phi(H_\phi^{-1}(t))\|\mathcal{P}_t(x,\cdot) - \pi(\cdot)\|_{TV} = 0.$$

No equivalence

- As opposed to the geometric case, there is no equivalence between conditions in the sub-geometric setting.
- Such equivalence is unlikely to hold in this form (the Jensen inequality is in the wrong direction).
- Can we change the conditions to obtain some form of equivalence ?

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In the geometric framework, the proof of equivalence uses the following stopping times: for some $r > 0$, a set C and $T \sim \mathcal{E}(1)$,

$$\tilde{\tau}_C^r := \inf\left\{ t > 0, \int_0^t 1_C(X_s) ds \geq \frac{T}{r} \right\}.$$

Some new conditions

Theorem

Consider $(X_t)_{t\geq 0}$ and ϕ, H_ϕ as before. We have equivalence between:
1. there exist a compact petite set C, $r>0$ s.t., for $T \sim \mathcal{E}(1)$,

$$\mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}_C^r)] < \infty \text{ for all } x \in E,$$

and this quantity is uniformly bounded on C.
2. $\exists C$ compact, petite on E, $\kappa, \eta > 0$ and $\psi : \mathbb{R}_+ \times E \to [1,\infty)$ continuous, $\uparrow$ in its first argument, such that (roughly)

$$(\partial_t + \mathcal{L})\psi(t,x) \leq \kappa H_\phi^{-1}(t) 1_C(x) - \phi(H_\phi^{-1}(t)).$$

Both conditions are implied by Condition 2 of the DFG's theorem. They imply the existence of an invariant $\pi \in \mathcal{P}(E)$ with

$$\forall x \in E, \lim_{t\to\infty} \phi(H_\phi^{-1}(t)) \|\mathcal{P}_t(x,\cdot) - \pi(\cdot)\|_{TV} = 0.$$

Long-Time Behavior of Kinetic Equations with Boundary Effects

Context

In Chapter 6, we add a linear collision operator $\mathcal{C}$. We consider the same boundary condition as before (with the accommodation coefficient $\alpha \in [0,1]$), where M is the wall Maxwellian.

$$\partial_t f + v \cdot \nabla_x f = \mathcal{C} f, \qquad \text{in } (0,\infty) \times G.$$

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With the hypothesis that we will introduce on $\mathcal{C}$, this models for instance the Boltzmann equation with or without cut-off or the Landau equation, in the linearized (close to equilibrium) regime. E.g., for Boltzmann

$$\mathcal{C} f(v) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B(|v - v_*|, \omega) \big(f' M'_* + M' f'_* - f M_* - M f_* \big) \, \mathrm{d}v_* \mathrm{d}\omega,$$

with $v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\omega$, $v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\omega$ the post-collisional velocities, B the collision kernel.

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$\rightarrow$ We know that the solution $f_t \rightarrow M$ as $t \rightarrow \infty$. What is the corresponding rate ?

Assumptions on $\mathcal{C}$

Let us introduce $L^2_v(M^{-1}) := \left\{ f : \mathbb{R}^d \to \mathbb{R} \,\middle|\, \int_{\mathbb{R}^d} f^2 M^{-1} \mathrm{d}v < +\infty \right\}$ endowed with $(f, g) := \int_{\mathbb{R}^d} f\, g\, M^{-1}\, \mathrm{d}v$ and the associated norm $\| \cdot \|$.

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1. We have $\ker(\mathcal{C}) = \mathrm{Span}\{M, v_1 M, \dots, v_d M, |v|^2 M\}$ on $L^2_v(M^{-1})$ (conservation laws) and we write πf for the projection of f on $\ker(\mathcal{C})$ and $f^\perp := f - \pi f$.

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2. The operator $\mathcal{C}$ is self-adjoint, with $(\mathcal{C} f, f) \leq 0$ and $\exists \lambda > 0$ s.t.

$$(-\mathcal{C} f, f) \geq \lambda \|f^\perp\|, \quad \forall f \in \mathrm{Dom}(\mathcal{C}).$$

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Let $\mathcal{L} = -v \cdot \nabla_x + \mathcal{C}$. We want something of the form

$$\langle -\mathcal{L}f, f \rangle \ge \lambda' |f|,$$

for some scalar product $\langle \cdot, \cdot \rangle$ and some $\lambda' > 0$.

Many results in the literature

Short-range interactions (Boltzmann with angular cutoff):

- Guo (2010): exponential convergence in weighted L^∞ space with specular reflection and diffuse reflection when Ω is strictly convex and analytic, see also Briant (2017) $\to L^2 - L^\infty$ techniques (non-constructive).
- Briant-Guo (2016): constructive results in L^2 if $\alpha > 0$ leading to exponential convergence in weighted L^∞ norm.
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Long-range interactions:

- Guo-Hwang-Jang-Ouyang (2020-2020): Landau equation with specular reflection, exponential convergence in L^2 norm.
- Duan-Liu-Sakamoto-Strain (2020): estimates in L^2 for non cut-off Boltzmann and Landau with specular reflection.

Hypocoercivity for linear equations

- Assume for simplicity that Ω has no rotational symmetries. Let $\mathcal{H} := \{f : G \to \mathbb{R}, \int_G f^2 M^{-1} \,\mathrm{d}v\mathrm{d}x < \infty\}$.

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- 2. To control the macroscopic part πf, we add new terms to this scalar product, and consider instead

$$\langle\langle f, g\rangle\rangle = \langle f, g\rangle - \epsilon\langle\bar{\pi} f, \nabla\Delta^{-1}\pi g\rangle_{L^2_x(\Omega)} - \epsilon\langle\nabla\Delta^{-1}\pi f, \bar{\pi} g\rangle_{L^2_x(\Omega)},$$

for $\epsilon > 0$ small enough and well-chosen operators $\bar{\pi}$. We want to obtain an equivalent scalar product (i.e. equivalent corresponding norms).

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  for $\epsilon > 0$ small enough and well-chosen operators $\bar{\pi}$. We want to obtain an equivalent scalar product (i.e. equivalent corresponding norms).
- One of the main difficulties: the Poisson equations associated with the Δ^{-1} have to be completed with adapted boundary conditions, to control the macroscopic quantities. This is especially difficult for the momentum component.

Results

Our results are **constructive** and treat the Maxwell boundary condition in **full generality**, i.e. $\alpha \in [0,1]$. Both Boltzmann (with and without cutoff) and Landau are handled.

Theorem (B., Carrapatoso, Mischler, Tristani)

Let $f_0 \in \mathcal{H}$ such that

- in the case $\alpha \equiv 0$: $\int_G f_0 \mathrm{d}v\mathrm{d}x = \int_G |v|^2 f_0 \mathrm{d}v\mathrm{d}x = 0$,
- otherwise: $\int_G f_0 \mathrm{d}v\mathrm{d}x = 0$.

There exist $\kappa, C > 0$ such that for all f solution with initial data f_0, for all $t \geq 0$,

$$\|f(t)\|_{\mathcal{H}} \leq Ce^{-\kappa t}\|f_0\|_{\mathcal{H}}.$$

Long-Time Behavior of Kinetic Equations with Boundary Effects

Outlook

- What about the Cercignani-Lampis boundary condition for the free-transport equation ? (in progress)
- Following the first point, what happens in collisional kinetic theory with this new boundary condition ?
- The stochastic process defined in Chapter 2 has recently been adapted in order to obtain simple NESS (non-equilibrium steady states). Can coupling methods (or sticky couplings, see EGZ), give results in this case ?
- Interactions of the new equivalent conditions with weak Poincaré/Cheeger's inequalities and sticky coupling. (in progress)
- Going beyond the L^2 case for the Maxwell boundary condition in collisional kinetic theory ?