Invariance principle for the random walk in random environment

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Donsker invariance principle. Let $(Y_k)_{k\geq 0}$ be i.i.d. random variables of mean 0 and variance 1. Once properly rescaled, the random walk $X_k = \sum_{\ell=1}^{k} Y_\ell$ behaves like a Brownian motion. Here, $\mathcal{D}(0,1)$ is the space of càdlàg functions from $[0,1]$ to $\mathbb{R}$ endowed with the Skorohod metric that turns it into a Polish metric space.

Theorem [Donsker's, 1950's]: Let, for all $t \in [0,1]$, $n \geq 1$, $X_t^n := \frac{X_{[n^2 t]}}{n}$. In $\mathcal{D}(0,1)$, we have convergence in the sense of distributions towards a standard Brownian motion with diffusion matrix I_d (we write sBM(I_d)) $(W_t)_{t\geq 0}$:

$$(X_t^n)_{t\in[0,1]} \xrightarrow{d} (W_t)_{t\in[0,1]}.$$

Let $(e_1,\dots,e_d)$ be the canonical basis of $\mathbb{R}^d$ and $E = \{e_1,\dots,e_d,-e_1,\dots -e_d\}$. On the grid $\mathbb{Z}^d$, we put random conductances: at each site $x \in \mathbb{Z}^d$, the conductance in direction $e \in E$ is given by $a_e(x)$ with $a_e(x)$ in $[b_1,b_2]$, $0 < b_1 < b_2 < \infty$ although those uniform ellipticity hypotheses can be relaxed. We impose a **symmetry condition**:

$$a_e(x) = a_{-e}(x+e), \qquad \forall x \in \mathbb{Z}^d, \quad \forall e \in E.$$

The random walk in random environment (RWRE) is a stochastic process (that is, a collection of random vectors indexed by $t \in \mathbb{R}_+$) denoted $(X_t)_{t\geq 0}$ that evolves as follows: assume at time $t \geq 0$, $X_t = x$ with $x \in \mathbb{Z}^d$. We consider $2d$ random clocks, τ_e with $e \in E$ s.t. τ_e has exponential distribution with mean $a_e(x)$. Then

$$X_{t+s} = X_t, \quad s \in [0, \min_{e\in E} \tau_e], \qquad X_{t+\min_{e\in E}\tau_e} = X_t + \big(\mathrm{argmin}_{e\in E}\tau_e\big)$$

i.e. the random walk moves in the direction indicated by the first ringing clock.

The increments of the RWRE are not i.i.d. because they depend on the conductances. Can we still derive an invariance principle for the RWRE ? There are two sources of randomness:

1. the environment itself, since the $a_e(x)$, $x \in \mathbb{Z}^d$, $e \in E$ are random.
2. The random walks performed on a given environment, that is the way the clocks described above ring (and how the RWRE thus behaves).

Quenched results describe the behavior of the random walk in a given environment $a := \{a_e(x), x \in \mathbb{Z}^d, e \in E\}$ in the space of environments $\mathbf{N}$ (see the next report), while *annealed results*, on which we focus here, consider a distribution μ on $\mathbf{N}$ and provide results for the annealed measure given, for any event A, by

$$\mathbb{P}(A) = \int_{\mathbf{N}} P^a(A)\mu(da).$$

Some notions about Markov processes. The RWRE is a Markov process, and the argument of [1] and [2] rely on an abstract result concerning those.

Let us introduce a bit of probabilistic machinery. Let $(Z_t)_{t\geq 0}$ be a stochastic process on $(\Omega,\mathcal{F},\mathbb{P})$, $\mathcal{F}_s = \sigma(Z_u, 0 \leq u \leq s)$ for all $s \geq 0$ so that $(\mathcal{F}_s)_{s\geq 0}$ is the canonical filtration of $(Z_t)_{t\geq 0}$. We say that $(Z_t)_{t\geq 0}$ is a Markov process if we have,

$$\forall A \in \mathcal{F}, \quad \mathbb{P}(Z_{t+s} \in A|\mathcal{F}_t) = \mathbb{P}(Z_{t+s} \in A|Z_t), \qquad t,s \geq 0.$$

In words, the information contained in $\mathcal{F}_t$ (i.e. everything that happened to the process up to time t) is exactly as relevant to predict the future value Z_{t+s} as the value Z_t itself, which is a priori a much smaller information. Markov processes appear in numerous contexts and share many key properties. We consider processes with values in a state space $(G, \mathcal{G})$ that are time-homogeneous. In particular one can identify a *transition kernel* or *transition semigroup* $(S^t)_{t\geq 0}$ such that

$$\forall B \in \mathcal{G}, \qquad S^t(z, B) = \mathbb{P}(Z_t \in B | Z_0 = z).$$

From there we can also introduce the *generator* of the process, which is formally $\mathcal{L} = \partial_t S^t_{|t=0}$ and the notion of stationary measure: if μ is a stationary measure for $(Z_t)_{t\geq 0}$, and if $Z_0 \sim \mu$ (i.e. Z_0 has distribution μ), then $Z_t \sim \mu$ for all $t \geq 0$. The precise formulation writes, for all $B \in \mathcal{G}$,

$$\int_G S^t(z, B)\mu(dx) = \mu(B).$$

The stationary measure μ is *ergodic* if for all $B \in \mathcal{G}$ such that $S^t(z, B) = 1$ for all $z \in B$, $\mu(B) \in \{0, 1\}$. This means that any absorbing set (i.e. a set that, if reached, captures the process forever) is either somewhere where the process spends all the time or no time when starting from the invariant measure. We will need the notion of reversibility: the process is as likely to go from z to y in a time $t > 0$ than it is to go from y to z. Mathematically, for all time $t > 0$

$$S^t(z, dy)\mu(dz) = S^t(y, dz)\mu(dy).$$

At last, roughly, a Markov process $(Z_t)_{t\geq 0}$ is a *martingale* if $Z_t \in L^1$ and if $\mathbb{E}[Z_t|\mathcal{F}_s] = Z_s$ almost surely, for all $t \geq s \geq 0$.

Theorem [1]: Let $(Z_t)_{t\geq 0}$ be a reversible Markov process with generator $\mathcal{L}$ and stationary measure μ, write $(\mathcal{F}_t)_{t\geq 0}$ for the corresponding filtration. Assume that μ is translation invariant and ergodic. Let X be a family indexed by closed bounded intervals of $\mathbb{R}$ with values in $\mathbb{R}^d$, anti-symmetric, i.e. if $I = [a, b]$,

$$X_I((Z_s)_{s\in I}) = -X_I((Z_{b+a-s})_{s\in I}).$$

Assume that the following strong L^1 limit exists,

$$\lim_{\delta\to 0} \tfrac{1}{\delta}\mathbb{E}_\mu\left[X_{[0,\delta]}\middle|\mathcal{F}_0\right] =: b(Z_0)$$

and that the martingale $M_t = X_t - \int_0^t b(Z_s)ds$ is square integrable. Defining $D_{ij} = C_{ij} + 2(b_i, \mathcal{L}^{-1}b_j)$, where C satisfies $e^T Ce = \mathbb{E}_\mu[(e \cdot M_1)^2]$, we have

$$\frac{1}{n}X_{[0,n^2t]} \to W_D,$$

in the sense of finite-dimensional distributions, where W_D is a sBM(D).

The idea here is that one can obtain an invariance principle when considering increments related to a reversible process having a stationary distribution that is mixing and translation invariant. The proof is based on [2] and is subtle: in particular showing that all quantities admitting a drift b so that M is a square-integrable martingale are such that $(b_i, \mathcal{L}^{-1}b_j)$ is well-defined is difficult. One can however relate to the usual central limit theorem: to be able to describe the behavior at the limit one needs a second moment. The condition on the drift here are analogous to this requirement.

An invariance principle for the RWRE. To conclude on the RWRE, it only remains to find such a reversible process $(Z_t)_{t\geq 0}$ and the appropriate additive functionals X. We introduce the process of the *environment seen by the particle* (see [2]). Keeping the particle centered at 0 along the walk, and translating the environment to compensate its jumps, how does the latter evolve ? We thus consider the process $(A_t)_{t\geq 0}$ with generator

$$\mathcal{L}f(a) = \sum_{e\in E} a_e(0)(f(S_{-e}a) - f(a))$$

i.e. at rate $a_e(0)$, the environment a is replaced by the one translated by $-e$: $(S_{-e}a)_{e'}(x) = a_{e'}(x+e)$ for all $x \in \mathbb{Z}^d$, $a \in \mathbf{N}$, $e, e' \in E$. If now μ is a distribution on $\mathbf{N}$ ergodic and translation invariant, satisfying the symmetry assumption, then the process $(A_t)_{t\geq 0}$ with $A_0 \sim \mu$ is stationary, ergodic and reversible. How to reconstruct the random walk from A? Coming back to the discrete setting, if $(Y_n)_{n\geq 0}$ denote the successive positions of the random walk, we can introduce a random time $n^*(t)$ such that $n^*(t) = n$ if $X_t = Y_n$, as well as a discrete environment process $(B_n)_{n\geq 0}$. One can easily reconstruct $(Y_n)_{n\geq 0}$ from $(B_n)_{n\geq 0}$: we have

$$Y_0 = 0, \qquad Y_{n+1} = Y_n + x \quad \text{if } B_{n+1} = S_{-x}B_n.$$

Once the chain $(Y_n)_{n\geq 0}$ is identified, one can rebuild $(X_t)_{t\geq 0}$ from it. The previous theorem then applies, and we find that for all bounded continuous functions $F \in \mathcal{D}(0,\infty)$ the Skorohod space, setting

$$(X^n_t)_{t\geq 0} := (\tfrac{1}{n}X_{n^2t})_{t\geq 0}, \qquad \text{one has} \quad \mathbb{E}_\mu[F(X^n)] \to \mathbb{E}[F(W_D)],$$

with W_D is a sBM(D), where, writing $\langle c\rangle_\mu = \int_{\mathbf{N}} c(a)\mu(da)$ for $c : \mathbf{N} \to \mathbb{R}$,

$$(1) \qquad D_{ij} = 2\Big\langle a_{e_i}(0)\delta_{ij}\Big\rangle_\mu + 2\Big\langle \big(a_{e_i}(0) - a_{e_i}(-e_i)\big)\mathcal{L}^{-1}\big(a_{e_j}(0) - a_{e_j}(-e_j)\big)\Big\rangle_\mu.$$

A comparison with the PDE viewpoint. In the PDE setting, focusing on the diagonal terms for simplicity, the homogenized coefficient $\bar{a}$ is given by

$$(2) \qquad \bar{a}e_i = \mathbb{E}[a(0)e_i] + \mathbb{E}\big[a(0)\nabla\phi_i(0)\big],$$

where ϕ is the corrector. With $\mathbf{L}$ the operator, the equation for the corrector writes $\mathbf{L}\phi_i = \mathrm{div}(ae_i)$. We can see here (see the discussion about the discrete gradient), that considering ∇ to be the discrete gradient instead, formally, we obtain $\phi_i(0) = \mathbf{L}^{-1}\mathrm{div}(a_{e_i}(0)) = -\mathbf{L}^{-1}(a_{e_i}(0) - a_{e_i}(-e_i))$. Coming back to (2), we get for the second term, using the translation invariance and as expected

$$\begin{aligned}\mathbb{E}[a(0)\nabla\phi_i(0)] &= \mathbb{E}\Big[\big(a_{e_i}(-e_i) - a_{e_i}(0)\big)\phi_i(0)\Big] \\ &= \mathbb{E}\Big[\big(a_{e_i}(0) - a_{e_i}(-e_i)\big)\mathbf{L}^{-1}\big(a_{e_i}(0) - a_{e_i}(-e_i)\big)\Big].\end{aligned}$$

References

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[2] C. Kipnis and S. R. S. Varadhan *Central Limit Theorem for Additive Functionals of Reversible Markov Processes and Applications to Simple Exclusions*, Communication in Mathematical Physics **104** (1986), 1-19.