Beyond propagation of chaos : correlations control in mean-field systems

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Correlations control in mean-field systems

Brownian particle system

Brownian particles: on the torus $\mathbb{T}^d$, for $t \geq 0, 1 \leq i \leq N$,

$$\begin{cases} Y_t^{i,N} = Y_0^{i,N} + \int_0^t \int_{\mathbb{T}^d} b(Y_t^{i,N} - z)\mu_s^N(\mathrm{d}z)\mathrm{d}s + B_t^i, \\ \mu_s^N := \frac{1}{N} \sum_{i=1}^N \delta_{Y_s^{i,N}}, \end{cases}$$

- $(Y^{i,N})_{i=1}^N$ positions in $\mathbb{T}^d$, $(Y_0^{i,N})_{1 \leq i \leq N} \overset{i.i.d.}{\sim} \mu_\circ$, $\int_{\mathbb{T}^d} |z|^{p_0} \mu_\circ < \infty$ for some $p_0 > 0$;
- μ_s^N is the empirical measure at time s;
- $b : \mathbb{T}^d \to \mathbb{R}^d$ is an interaction potential;
- mean-field scaling.

$\to$ We'll take $b(x-z) = -\kappa \nabla W(x-z)$ for a *smooth* W with $W(x) = W(-x)$ and $\kappa > 0$ small.

Langevin particle system

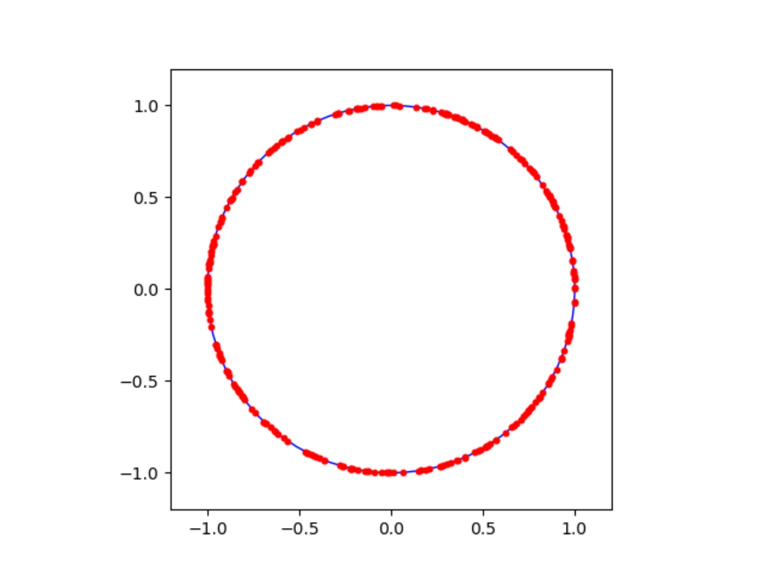
Phase space $\mathbb{T}^d \times \mathbb{R}^d$, for $t \geq 0, 1 \leq i \leq N$,

$$\begin{cases} X_t^{i,N} = X_0^{i,N} + \int_0^t V_s^{i,N} \mathrm{d}s, \\ V_t^{i,N} = V_0^{i,N} - \frac{\beta}{2} \int_0^t V_s^{i,N} \mathrm{d}s + \int_{\mathbb{T}^d} b(X_t^{i,N} - z)(\mu_x^N)_s(\mathrm{d}z)\mathrm{d}s + B_t^i, \\ \mu_s^N := \frac{1}{N} \sum_{i=1}^N \delta_{X_s^{i,N}, V_s^{i,N}}, \qquad (\mu_x^N)_s = \frac{1}{N} \sum_{i=1}^N \delta_{X_s^{i,N}}. \end{cases}$$

- $(X^{i,N})_{i=1}^N$ positions in $\mathbb{T}^d$, $(V^{i,N})_{i=1}^N$ velocities in $\mathbb{R}^d$;
- μ_s^N is the empirical measure at time s, $(\mu_x^N)_s$ empirical measure of positions at time s;
- $b : \mathbb{T}^d \to \mathbb{R}^d$ is an interaction potential (only uses positions);
- $\mu_\circ$ define on $\mathbb{T}^d \times \mathbb{R}^d$ with some moment.
- the mean-field scaling is considered.

$\to$ talk will be given for Brownian particles, but only one major difference between both.

Brownian particles with mean-field interaction



Natural questions

1. Law of large numbers ? Called here, **propagation of chaos**, typical behavior of one particle as $N \to \infty$.
2. Central Limit Theorem ? Scaling and description of the fluctuations around this limit equation.
3. Concentration estimates ?
4. **Refined** propagation of chaos ? Corrections to the mean-field limit.

In what sense ?

A complete answer to all of those would include a **time-uniform** statement: the associated errors do not deteriorate with time.

Density, marginal distributions

F^N probability density of the system in $(\mathbb{T}^d)^N$. Solves a forward Kolmogorov equation, where $\bar{x} = (x_1, \dots, x_N) \in (\mathbb{T}^d)^N$

$$\partial_t F^N(\bar{x}) = \frac{1}{2}\triangle F^N(\bar{x}) + \kappa \sum_{i=1}^N \mathrm{div}_{x_i}\Big(F^N(\bar{x})\frac{1}{N}\sum_{j=1}^N \nabla W(x_j - x_i)\Big).$$

Marginal distribution of $k \geq 1$ particles:

$$F_N^k(t, x_1, \dots, x_k) = \int_{(\mathbb{T}^d)^{N-k}} F^N(t, \bar{x})\,\mathrm{d}x_{k+1}\dots\mathrm{d}x_N.$$

Correlations:

1. 2 particles: $G_N^2(x_1, x_2) = F_N^2(x_1, x_2) - F_N^1(x_1)F_N^1(x_2)$,
2. 3 particles:

$$G_N^3(x_1, x_2, x_3) = \mathsf{Sym}\big(F_N^3 - 3F_N^2 \otimes F_N^1 + 2(F_N^1)^{\otimes 3}\big).$$

3. and so on...

Observables, correlations

Note, if $\varphi : \mathbb{T}^d \to \mathbb{R}$ bounded,

$$\mathbb{E}\Big[\int_{\mathbb{T}^d} \varphi(x)\mu_t^N(\mathrm{d}x)\Big] = \int_{\mathbb{T}^d} \varphi(x) F_N^1(t,x)\mathrm{d}x.$$

This talk $\to$ at the level of **observables**: the behavior of the random variable $\int_{\mathbb{T}^d} \varphi(x)\mu_t^N(\mathrm{d}x)$ for φ smooth.

Correlations studied through **cumulants** of the observable: e.g.

$$\begin{aligned}\kappa^2\Big[\int_{\mathbb{T}^d} \varphi\mu_t^N\Big] &= \mathbb{E}\Big[\Big(\int_{\mathbb{T}^d} \varphi\mu_t^N\Big)^2\Big] - \mathbb{E}\Big[\int_{\mathbb{T}^d} \varphi\mu_t^N\Big]^2 \\ &= \frac{1}{N}\mathrm{Var}[\varphi(Y^{1,N})] \\ &\quad + \frac{N-1}{N}\int_{(\mathbb{T}^d)^2} \varphi(x_1)\varphi(x_2) G_N^2(x_1,x_2)\mathrm{d}x_1\,\mathrm{d}x_2,\end{aligned}$$

and in general

$$\kappa^m\Big[\int_{\mathbb{T}^d} \varphi\mu_t^N\Big] = \int_{(\mathbb{T}^d)^m} \varphi^{\otimes m} G_N^m \mathrm{d}x_1 \dots d_{x_m} + O\Big(\frac{1}{N^m}\Big).$$

Correlations control in mean-field systems

Propagation of chaos

Integrating the Kolmogorov equation and using symmetries

$$\partial_t F_N^1(x) = \frac{1}{2}\triangle F_N^1(x) + \kappa\,\mathrm{div}_x\Big(\int_{\mathbb{T}^d} \nabla W(y-x) F_N^2(x,y)\mathrm{d}y\Big).$$

Propagation of chaos: since $F_N^2 = F_N^1 \otimes F_N^1 + G_N^2$ and if $G_N^2 \to 0$ as $N \to \infty$, we get the limit equation

$$\partial_t f(t,x) = \frac{1}{2}\triangle f(x) + \kappa\,\mathrm{div}_x\Big(f(x)\int_{\mathbb{T}^d} \nabla W(x-y)\, f(y)\,\mathrm{d}y\Big).$$

Also true as an evolution in the space of measures:

$$\partial_t m(t,\mu) = \tfrac{1}{2}\triangle m(t,\mu) + \kappa\,\mathrm{div}_x\Big(m(t,\mu)\int_{\mathbb{T}^d} \nabla W(x-y)\, m(t,\mu)(\mathrm{d}y)\Big)$$

$$m(0,\mu) = \mu.$$

Hence, we expect

$$\mathbb{E}\Big[\int_{\mathbb{T}^d} \varphi(x)\mu_t^N(\mathrm{d}x)\Big] = \int_{\mathbb{T}^d} \varphi(x) F_N^1(t,x)\mathrm{d}x \approx \int_{\mathbb{T}^d} \varphi(x) m(t,\mu_\circ)(\mathrm{d}x).$$

Quantitative propagation of chaos: weak form

See Chaintron-Diesz (2023) for more ref. and other notions of propagation of chaos. Recall $\mathbb{E}[\Phi(\mu_t^N)] = \int_{\mathbb{T}^d} \varphi(x) F_N^1(\mathrm{d}x)$.
From Mischler-Mouhot, Delarue-Tse, Jourdain...

$$\mathbb{E}[\Phi(\mu_t^N)] - \Phi(m(t, \mu_\circ) \leq \tilde{\theta}(N, t),$$

with $\tilde{\theta}(N, t) \to 0$ as $N \to \infty$, typically of order N^{-1}.

Key questions: $\sup_{t \geq 0} \tilde{\theta}(N, t) \leq \Theta(N)$?

Delarue-Tse (2021): under regularity assumptions on b and Φ, there exists $C > 0$ such that

$$\sup_{t \geq 0} \mathbb{E}\Big[\big|\Phi(\mu_t^N) - \Phi(m(t, \mu_\circ))\big|\Big] \leq \frac{C}{N}.$$

So in our setting, for this weak notion, Question 1 solved "completely".

Natural questions, v2

For $\varphi : \mathbb{T}^d \to \mathbb{R}$ smooth, $\Phi(\mu) := \int_{\mathbb{T}^d} \varphi(x)\mu(\mathrm{d}x)$,

1. Law of large numbers (Delarue-Tse)

$$\mathbb{E}\big[\Phi(\mu_t^N)\big] - \Phi(m(t,\mu_\circ)) \leq \frac{C}{N}$$

2. Central Limit Theorem: existence of some $(\nu_t)_{t \geq 0}$ s. t.

$$d\Big(\sqrt{N}\int_{\mathbb{T}^d} \varphi(x)\big(\mu_t^N - m(t,\mu_\circ)\big)(\mathrm{d}x), \int_{\mathbb{T}^d} \varphi \nu_t\Big) \leq \theta_2(N,t)$$

3. Concentration: for some $C > 0$, for $r > 0$ (conditions on r ?)

$$\mathbb{P}\Big[\Big|\int_{\mathbb{T}^d} \varphi(x)\big(\mu_t^N - m(t,\mu_\circ)\big)(\mathrm{d}x)\Big| \geq r\Big] \leq e^{-CNr^2}$$

4. **Refined** propagation of chaos ? Corrections to the mean-field limit

$$G_t^{m,N} \leq CN^{1-m}, \qquad \kappa^m[\Phi(\mu_t^N)] \leq CN^{1-m}$$

for all $m \geq 2$? In what sense for the first inequality ?

And can we get uniform-in-time results for 2-4 ?

Corrections the limit equation

What if we know that G_N^2 is of order $O\left(\frac{1}{N}\right)$?

For F_N^1, we can also get

$$\partial_t F_N^1(x) = \frac{1}{2}\triangle F_N^1(x) + \kappa\,\mathrm{div}_x\Big(F_N^1(x)\int_{\mathbb{T}^d}\nabla W(x-y)\,F_N^1(y)\,\mathrm{d}y\Big)$$
$$+\,\kappa\,\mathrm{div}_x\Big(\frac{1}{N}\int_{\mathbb{T}^d}\nabla W(x-y)(NG_N^2)(x,y)\mathrm{d}y\Big)$$

Corrections to the limit equation II

Assuming that $G_N^3 = O\Big(\frac{1}{N^2}\Big)$ $\to$ evolution equation on F_N^2

$$\begin{aligned}\partial_t F_N^2(x_1,x_2) = \frac{1}{2}\triangle F_N^2(x_1,x_2) - \kappa \sum_{1\leq i\neq j\leq 2} \mathrm{div}_{x_i}\Big\{ -\frac{1}{N}\nabla W(x_i-x_j)F_N^1(x_i)F_N^1(x_j)\\ + \frac{N-1}{N} b(x_i,F_N^1)F_N^1(x_i)F_N^1(x_j) + 3\frac{N-1}{N} b(x_i,F_N^1)F_N^2(x_i,x_j)\\ - 3\,\kappa\,\frac{N-1}{N} F_N^1(x_j)\int_{\mathbb{T}^d} \nabla W(x-x_i)F_N^2(x_i,x)\mathrm{d}x\\ - 3\,\kappa\,\frac{N-1}{N} F_N^1(x_1)\int_{\mathbb{T}^d} \nabla W(x-x_i)F_N^2(x,x_j)\mathrm{d}x\Big\} + O\Big(\frac{1}{N^2}\Big).\end{aligned}$$

Using that $G_N^2 = F_N^2 - (F_N^1)^{\otimes 2}$ we get a closed form for the evolution of F_N^1 and G_N^2. The initial data are $G^2_{N|t=0} = 0$, $F^1_{N|t=0} = \mu_\circ$.

Correlations control in mean-field systems

Uniform-in-time control of correlations

Theorem (B.-Duerinckx, 2024)

There exists $\kappa_0 > 0$ *such that for any* $\kappa \in [0, \kappa_0)$, *for all* $2 \leq m \leq N$, *there exists* $\ell_m > 0$, $C_m > 0$ *(only depending on* $d, \beta, W, \mu_\circ, m$*) such that, for all* $t \geq 0$,

$$\|G_N^m(t)\|_{W^{-\ell_m,1}(\mathbb{T}^d)} \leq C_m N^{1-m}.$$

$\rightarrow$ uniform-in-time answer to Question 4. Combined with Herbst's argument: concentration estimates, uniform-in-time answer to Question 3.

Hess-Childs, Rowan 2023: similar result for Brownian systems, non-uniform in time but with stronger norms, through hierarchical methods.

Further results: uniform-in-time CLT

Theorem (B.-Duerinckx, 2024)

There exists $\kappa_0, \lambda_0 > 0$ *such that for any* $\kappa \in [0, \kappa_0)$, *for* $\mu_t = m(t, \mu_\circ)$, *for all* $\varphi \in C_c^\infty(\mathbb{T}^d)$, *there exists* $C_\varphi > 0$ *such that for all* $N, t \geq 0$,

$$d_2\Big(\sqrt{N}\Big(\int_{\mathbb{T}^d} \varphi \mu_t^N - \int_{\mathbb{T}^d} \varphi\, \mu_t\Big), \int_{\mathbb{T}^d} \varphi \nu_t\Big) \leq C_\phi\Big(N^{-\frac{1}{2}} + e^{-p_0 \lambda_0 t} N^{-\frac{1}{3}}\Big),$$

where d_2 *is the Zolotarev distance, and where* $(\nu_t)_{t\geq 0}$ *solves the Gaussian linearized Dean-Kawasaki SPDE*

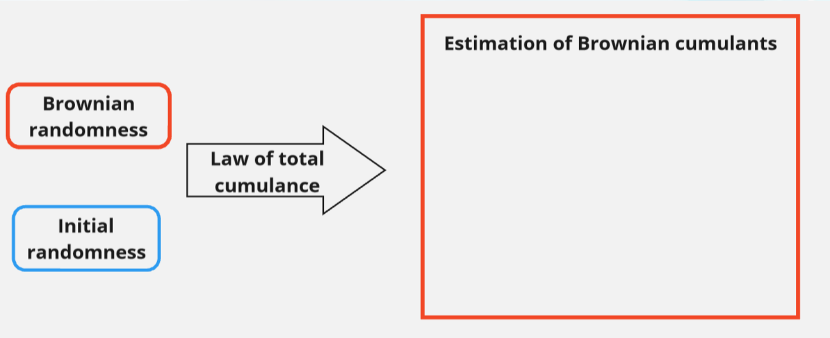
$$\begin{cases} \partial_t \nu_t + v \cdot \nabla_x \nu_t &= \operatorname{div}_v(\sqrt{\mu_t}\xi_t) + \operatorname{div}_v\Big(\big(\nabla_v + \beta v\big)\nu_t\Big) \\ & \quad + \kappa(\nabla W \star \nu_t) \cdot \nabla_v \mu_t + \kappa(\nabla W \star \mu_t) \cdot \nabla_v \nu_t, \\ (\nu_t)_{|t=0} = \nu_\circ, \end{cases}$$

where ξ *is a space-time white noise in* $\mathbb{R}_+ \times \mathbb{T}^d$ *and for all* $\varphi \in C_c^\infty(\mathbb{T}^d)$

$$\sqrt{N} \int_{\mathbb{T}^d} \varphi(\mu_0^N - \mu_\circ) \xrightarrow{\mathcal{L}} \int_{\mathbb{T}^d} \varphi \nu_\circ.$$

Correlations control in mean-field systems

Two sources of randomness



$\Phi(\mu) = \int_{\mathbb{T}^d} \varphi(x)\mu(\mathrm{d}x)$, $\varphi : \mathbb{T}^d \to \mathbb{R}$ smooth. We want $\kappa^2\left[\Phi(\mu_t^N)\right] = O\left(\frac{1}{N}\right)$.

$$\mathrm{Var}[\Phi(\mu_t^N)] = \mathrm{Var}_\circ\left[\mathbb{E}_B[\Phi(\mu_t^N)]\right] + \mathbb{E}_\circ\left[\mathrm{Var}_B[\Phi(\mu_t^N)]\right] = O\left(\frac{1}{N}\right) \qquad ?$$

Same type of decomposition for all cumulants.

Lions expansion along the flow (CST 22, DT 21)

Let $\mathcal{U}_\Phi\big((t,s),\mu\big) = \Phi(m(t-s,\mu))$. Then,

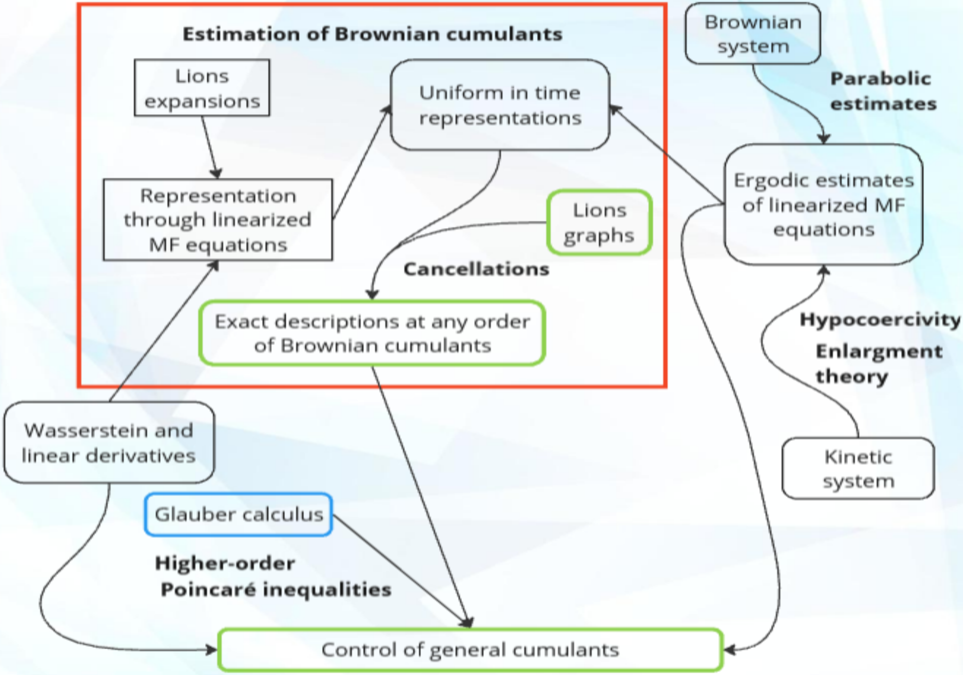
$$\begin{aligned}\mathbb{E}_B\Big[\Phi(\mu_t^N)\Big] &= \mathbb{E}_B\Big[\mathcal{U}_\Phi\big((t,t),\mu_t^N\big)\Big] \\ &= \Phi(m(t,\mu_0^N)) \\ &\qquad + \underbrace{\frac{1}{2N}\int_0^t \mathbb{E}_B\Big[\int_{\mathbb{T}^d} \mathrm{Tr}\big[\partial_\mu^2 \mathcal{U}_\Phi\big((t,s),\mu_s^N\big)(v,v)\big]\mu_s^N(\mathrm{d}v)\Big]\mathrm{d}s}_{\substack{=\frac{1}{2N}\int_0^t\int_{\mathbb{T}^d}\mathrm{Tr}\left[\partial_\mu^2\mathcal{U}_\Phi\left((t,s),m(s,\mu_0^N)\right)(v,v)\right]m(s,\mu_0^N)(\mathrm{d}v)\mathrm{d}s \\ +\text{Terms in }\frac{1}{N^2}}}\end{aligned}$$

This can be used to expand $\mathbb{E}_B[\Phi^2(\mu_t^N)]$ as well ! Then

$$\mathrm{Var}_B[\Phi(\mu_t^N)] = \mathbb{E}_B[\Phi^2(\mu_t^N)] - \mathbb{E}_B[\Phi(\mu_t^N)]^2$$

The terms of order 1 cancel out !

Key point: we have representation formula for $\partial_\mu^2 \mathcal{U}_\Phi$ using linearized evolutions. We can **truncate expansions uniformly in time**.


Estimation of Brownian cumulants
Lions expansions
Uniform in time representations
Representation through linearized MF equations
Lions graphs
Cancellations
Exact descriptions at any order of Brownian cumulants
Brownian system
Parabolic estimates
Ergodic estimates of linearized MF equations
Hypocoercivity
Enlargment theory
Kinetic system
Wasserstein and linear derivatives
Glauber calculus
Higher-order Poincaré inequalities
Control of general cumulants

Thanks for listening !

B.-Duerinckx, *Uniform-in-time estimates on the size of chaos for interacting Brownian particles*, arXiv 2405.19306.