# Beyond the mean-field limit for the McKean-Vlasov particle system: Uniform in time estimates for the cumulants

Armand Bernou (LJLL, Sorbonne Université)
Joint work with Mitia Duerinckx (FNRS)

Metastability, mean-field particle systems and non linear processes, Saint-Étienne

20th May, 2022

# Beyond the mean-field limit for the MV particle system

# The particle system: first-order, MF interaction and noise

We consider the following particle system, with particles living in $\mathbb{T}^d$:

$$\begin{cases} Y_t^{i,N} = Y_0^{i,N} + \int_0^t b(Y_s^{i,N}, \mu_s^N)\mathrm{d}s + B_t^i, & t \geq 0, 1 \leq i \leq N, \\ \mu_s^N := \frac{1}{N}\sum_{i=1}^N \delta_{Y_s^{i,N}}, \end{cases}$$

where $(Y^{i,N})_{i=1}^N$ denotes the set of positions of the particles in the $d$-dimensional torus, $\mu_s^N$ is the empirical measure at time $s$, $b : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}^d$ is an interaction potential and the mean-field scaling is considered.

## The particle system: first-order, MF interaction and noise

We consider the following particle system, with particles living in $\mathbb{T}^d$:

$$\begin{cases} Y_t^{i,N} = Y_0^{i,N} + \int_0^t b(Y_s^{i,N}, \mu_s^N)\mathrm{d}s + B_t^i, & t \geq 0, 1 \leq i \leq N, \\ \mu_s^N := \frac{1}{N} \sum_{i=1}^N \delta_{Y_s^{i,N}}, & \end{cases}$$

where $(Y^{i,N})_{i=1}^N$ denotes the set of positions of the particles in the $d$-dimensional torus, $\mu_s^N$ is the empirical measure at time $s$, $b : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}^d$ is an interaction potential and the mean-field scaling is considered.

In this talk: we live on the torus, with a smooth $b$. Also, $(Y_0^{i,N})_{1 \leq i \leq N} \sim \mu_0^{\otimes N}$, i.e. we have i.i.d. initial distributions.

# Assumptions on $b$

We place ourselves in the framework of Carrillo-Gvanali-Pavliotis-Schlichting (2019) by considering an H-stable potential , i.e. we assume

$$b(x,m) = -\kappa \int_{\mathbb{T}^d} \nabla W(x-y)\, m(\mathrm{d}y), \qquad x \in \mathbb{T}^d, m \in \mathcal{P}(\mathbb{T}^d)$$

for $\kappa > 0$ (equal to 1 in what follows for simplicity) and $W$ smooth, coordinate-wise even:

$$W(x_1, \cdots, -x_i, \dots, x_d) = W(x_1, \dots, x_i, \dots, x_d), \quad (x_1, \dots, x_d) \in \mathbb{T}^d.$$

Writing $(\hat{W}^n)_{n\in\mathbb{Z}^d}$ for the Fourier coefficients of $W$, we assume that, for any $n \in \mathbb{Z}^d$, $\hat{W}^n \geq 0$.

# Assumptions on $b$

We place ourselves in the framework of Carrillo-Gvanali-Pavliotis-Schlichting (2019) by considering an H-stable potential , i.e. we assume

$$b(x,m) = -\kappa \int_{\mathbb{T}^d} \nabla W(x-y)\, m(\mathrm{d}y), \qquad x \in \mathbb{T}^d, m \in \mathcal{P}(\mathbb{T}^d)$$

for $\kappa > 0$ (equal to 1 in what follows for simplicity) and $W$ smooth, coordinate-wise even:

$$W(x_1, \cdots, -x_i, \ldots, x_d) = W(x_1, \ldots, x_i, \ldots, x_d), \quad (x_1, \ldots, x_d) \in \mathbb{T}^d.$$

Writing $(\hat{W}^n)_{n \in \mathbb{Z}^d}$ for the Fourier coefficients of $W$, we assume that, for any $n \in \mathbb{Z}^d$, $\hat{W}^n \geq 0$.

This implies in particular that the free energy is convex (André's and Greg's talks) and the Lebesgue measure on $\mathbb{T}^d$, $\mathrm{Leb}_{\mathbb{T}^d}$ is the unique invariant measure, and is exponentially stable, in the sense that there exists $C, \lambda > 0$ constants s.t. for all $t \geq 0$,

$$\|m(t,\mu) - \mathrm{Leb}_{\mathbb{T}^d}\|_{TV} \leq Ce^{-\lambda t},$$

where $(m(t,\mu))_{t \geq 0}$ is the flow of marginal law.

## Statistical description and mean-field limit

Consider the regime $N \gg 1$. Let $F_N$ be the probability density on the $N$ torus $(\mathbb{T}^d)^N$ and $F_N^1$ be the first marginal

$$F_N^1(z) = \int_{(\mathbb{T}^d)^{N-1}} F_N(z, z_2, \dots, z_N) \mathrm{d}z_2 \dots \mathrm{d}z_N.$$

# Statistical description and mean-field limit

Consider the regime $N \gg 1$. Let $F_N$ be the probability density on the $N$ torus $(\mathbb{T}^d)^N$ and $F_N^1$ be the first marginal

$$F_N^1(z) = \int_{(\mathbb{T}^d)^{N-1}} F_N(z, z_2, \dots, z_N) \mathrm{d}z_2 \dots \mathrm{d}z_N.$$

We expect, when $N$ becomes large, and in view of **Boltzmann chaos assumption** to be able to neglect the correlations and to obtain, in the limit $N \to \infty$, that $F_N^1$ behaves like the solution of the McKean-Vlasov SDE:

$$X_t = X_0 + \int_0^t b\big(X_s, \mathcal{L}(X_s)\big) \mathrm{d}s + W_t, \qquad t \geq 0$$

where $\mathcal{L}(Z)$ denotes the law of $Z$ and $\mathcal{L}(X_0) = \mu_0$.

# Statistical description and mean-field limit

Consider the regime $N \gg 1$. Let $F_N$ be the probability density on the $N$ torus $(\mathbb{T}^d)^N$ and $F_N^1$ be the first marginal

$$F_N^1(z) = \int_{(\mathbb{T}^d)^{N-1}} F_N(z, z_2, \ldots, z_N) \mathrm{d}z_2 \ldots \mathrm{d}z_N.$$

We expect, when $N$ becomes large, and in view of **Boltzmann chaos assumption** to be able to neglect the correlations and to obtain, in the limit $N \to \infty$, that $F_N^1$ behaves like the solution of the McKean-Vlasov SDE:

$$X_t = X_0 + \int_0^t b\big(X_s, \mathcal{L}(X_s)\big)\mathrm{d}s + W_t, \qquad t \geq 0$$

where $\mathcal{L}(Z)$ denotes the law of $Z$ and $\mathcal{L}(X_0) = \mu_0$.

This is the key question of propagation of chaos. Question: what are the equations governing the corrections to the mean-field equation ? Can we justify them ?

## A PDE for the flow of marginal laws

Consider again the McKean-Vlasov SDE

$$X_t = X_0 + \int_0^t b\big(X_s, \mathcal{L}(X_s)\big)\mathrm{d}s + W_t, \qquad t \geq 0$$

with $\mathcal{L}(X_0) = \mu_0$ for some distribution $\mu_0$ on the torus. Starting from $\mu \in \mathcal{P}(\mathbb{T}^d)$, the flow of marginal laws $(m(t, \mu_0) := \mathcal{L}(X_t))_{t \geq 0}$ satisfies a nonlinear Fokker-Planck equation:

$$\partial_t m(t, \mu) = \frac{1}{2} \triangle m(t, \mu) - \operatorname{div}\Big[m(t, \mu) b(\cdot, m(t, \mu))\Big], \qquad t \geq 0,$$
$$m(0, \mu) = \mu,$$

at least in a distributional sense.

# BBGKY, MV version: a formal expansion

The phase space marginal $F^N$ solves a forward Kolmogorov equation, where $\bar{x} = (x_1, \dots, x_N) \in (\mathbb{T}^d)^N$

$$\partial_t F^N(\bar{x}) = \frac{1}{2} \triangle F^N(\bar{x}) + \sum_{i=1}^{N} \mathrm{div}_{x_i} \Big( F^N(\bar{x}) \frac{1}{N} \sum_{j=1}^{N} \nabla W(x_j - x_i) \Big).$$

# BBGKY, MV version: a formal expansion

The phase space marginal $F^N$ solves a forward Kolmogorov equation, where $\bar{x} = (x_1, \ldots, x_N) \in (\mathbb{T}^d)^N$

$$\partial_t F^N(\bar{x}) = \frac{1}{2} \triangle F^N(\bar{x}) + \sum_{i=1}^{N} \mathrm{div}_{x_i} \Big( F^N(\bar{x}) \frac{1}{N} \sum_{j=1}^{N} \nabla W(x_j - x_i) \Big).$$

Integrating this equation with respect to $x_2, \ldots, x_N$, writing $F_N^k$ for the $k$-th marginal, and using the coordinate-wise symmetry we get

$$\partial_t F_N^1(x) = \frac{1}{2} \triangle F_N^1(x) + \mathrm{div}_x \Big( \int_{\mathbb{T}^d} \nabla W(y - x) F_N^2(x, y) \mathrm{d}y \Big).$$

## BBGKY, MV version: a formal expansion

The phase space marginal $F^N$ solves a forward Kolmogorov equation, where $\bar{x} = (x_1, \dots, x_N) \in (\mathbb{T}^d)^N$

$$\partial_t F^N(\bar{x}) = \frac{1}{2}\triangle F^N(\bar{x}) + \sum_{i=1}^N \mathrm{div}_{x_i}\Big(F^N(\bar{x})\frac{1}{N}\sum_{j=1}^N \nabla W(x_j - x_i)\Big).$$

Integrating this equation with respect to $x_2, \dots, x_N$, writing $F_N^k$ for the $k$-th marginal, and using the coordinate-wise symmetry we get

$$\partial_t F_N^1(x) = \frac{1}{2}\triangle F_N^1(x) + \mathrm{div}_x\Big(\int_{\mathbb{T}^d} \nabla W(y-x) F_N^2(x,y)\mathrm{d}y\Big).$$

Propagation of chaos says that in the decomposition $F_N^2(x,y) = F_N^1(x)F_N^1(y) + G_N^2(x,y)$ where $G_N^2$ is the two-particles correlation function, $G_N^2 \to 0$ as $N \to \infty$. This leads to the PDE version of the McKean-Vlasov equation on $\mathbb{T}^d$

$$\partial_t f(x) = \frac{1}{2}\triangle f(x) - \mathrm{div}_x\Big(b\big(x, f\big)f(x)\Big).$$

## Beyond mean-fields (Bogolyubov corrections?)

What happens if we know that $G_N^2$ is of order $O\left(\frac{1}{N}\right)$ ? Coming back to the equation for $F_N^1$ depending on $F_N^2$, we get instead

$$\begin{aligned}\partial_t F_N^1(x) &= \frac{1}{2}\triangle F_N^1(x) - \mathrm{div}_x\big(b(x,F_N^1)F_N^1(x)\big)\\ &\quad + \mathrm{div}_x\left(\frac{1}{N}\int_{\mathbb{T}^d}\nabla W(x-y)(NG_N^2)(x,y)\mathrm{d}y\right)\end{aligned}$$

and, assuming that $G_N^3 = O\left(\frac{1}{N^2}\right)$, we have an evolution equation on $F_N^2$ given by

# Beyond mean-fields (Bogolyubov corrections?)

What happens if we know that $G_N^2$ is of order $O\left(\frac{1}{N}\right)$ ? Coming back to the equation for $F_N^1$ depending on $F_N^2$, we get instead

$$\partial_t F_N^1(x) = \frac{1}{2}\triangle F_N^1(x) - \mathrm{div}_x\left(b(x, F_N^1)F_N^1(x)\right) + \mathrm{div}_x\left(\frac{1}{N}\int_{\mathbb{T}^d} \nabla W(x-y)(NG_N^2)(x,y)\mathrm{d}y\right)$$

and, assuming that $G_N^3 = O\left(\frac{1}{N^2}\right)$, we have an evolution equation on $F_N^2$ given by

$$\begin{aligned}\partial_t F_N^2(x_1,x_2) = \frac{1}{2}\triangle F_N^2(x_1,x_2) - \sum_{1\le i\neq j\le 2} \mathrm{div}_{x_i}\Big\{ &-\frac{1}{N}\nabla W(x_i-x_j)F_N^1(x_i)F_N^1(x_j) \\ &+ \frac{N-1}{N} b(x_i, F_N^1)F_N^1(x_i)F_N^1(x_j) + 3\frac{N-1}{N} b(x_i, F_N^1)F_N^2(x_i,x_j) \\ &- 3\frac{N-1}{N}\int_{\mathbb{T}^d} \nabla W(x-x_i)F_N^2(x_i,x)\mathrm{d}x F_N^1(x_j) \\ &- 3\frac{N-1}{N}\int_{\mathbb{T}^d} \nabla W(x-x_i)F_N^2(x,x_j)\mathrm{d}x F_N^1(x_1)\Big\} + O\left(\frac{1}{N^2}\right).\end{aligned}$$

Using that $G_N^2 = F_N^2 - (F_N^1)^{\otimes 2}$ we get a closed form for the evolution of $F_N^1$ and $G_N^2$. The initial data are $G_{N,|t=0}^2 = 0$, $F_{N|t=0}^1 = \mu_0$.

# Beyond the mean-field limit for the MV particle system

# Propagation of chaos I: qualitative convergence

In general, we expect that on a finite time interval $[0, T]$, $T > 0$, and for any fixed $k \in \{1, \ldots, N\}$,

$$(Y^{1,N}, \ldots, Y^{k,N}) \Longrightarrow (X^1, \ldots, X^k),$$

where the $(X^i)_i$ are i.i.d. copies of solutions to the McKean-Vlasov equation, and where the convergence holds weakly in $C([0, T], (\mathbb{T}^d)^k)$. This gives both the convergence towards the limit equation and the asymptotic independence.

## Propagation of chaos I: qualitative convergence

In general, we expect that on a finite time interval $[0,T]$, $T>0$, and for any fixed $k \in \{1,\dots,N\}$,

$$(Y^{1,N},\dots,Y^{k,N}) \Longrightarrow (X^1,\dots,X^k),$$

where the $(X^i)_i$ are i.i.d. copies of solutions to the McKean-Vlasov equation, and where the convergence holds weakly in $C([0,T],(\mathbb{T}^d)^k)$. This gives both the convergence towards the limit equation and the asymptotic independence.

- For $b$ Lipschitz (w.r.t. the topology of $\mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$), this can be proved using the famous Sznitman's coupling, see Sznitman (1991), Lacker (2018)...
- Other approaches are based on the tightness of $(\mathcal{L}(\mu_t^N) \in \mathcal{P}(\mathcal{P}(\mathbb{T}^d)))_{0\le t\le T}$. One then shows that $(\mathcal{L}(\mu_t^N))_{0\le t\le T}$ converges weakly to $\delta_{\mathcal{L}(X_t)_{0\le t\le T}}$.

# Propagation of chaos II: strong errors

One way to divide the results on the propagation of chaos is to distinguish between strong and weak errors. By strong errors, I mainly mean convergence results expressed in some Wasserstein norm.
For instance, from Sznitman, with our choice of $b$, one can use coupling and then

$$\sup_{t\geq 0} W_2\Big(\mathcal{L}(Y_t^{i,N}), \mathcal{L}(X_t)\Big) = O\Big(\frac{1}{N^{\frac{1}{2}}}\Big).$$

# Propagation of chaos II: strong errors

One way to divide the results on the propagation of chaos is to distinguish between strong and weak errors. By strong errors, I mainly mean convergence results expressed in some Wasserstein norm.
For instance, from Sznitman, with our choice of $b$, one can use coupling and then

$$\sup_{t\geq 0} W_2\Big(\mathcal{L}(Y_t^{i,N}), \mathcal{L}(X_t)\Big) = O\Big(\frac{1}{N^{\frac{1}{2}}}\Big).$$

For the toroidal setting, more uniform in times results have (more or less recently) been derived. See in particular

- Malrieu 2001 (convex setting)
- Durmus-Eberle-Guillin-Zimmer 2020 for the case where the interaction is small
- Guillin-Le Bris-Monmarché 2021 for more singular interactions (allowing to treat the Biot-Savart kernel).

Jabin-Wang (2018), gives non-uniform in time estimates when the interaction is singular but several papers starts from their strategy (Arnaud's talk).
The strong error can also be quantified through a central limit theorem (Sznitman, Méléard...).

# Propagation of chaos III: weak errors

Weak errors are about the statistical behavior of the empirical distribution. Here, the focus is on deriving rates of convergence (in $t$ and $N$) for

$$\mathbb{E}\left[\left|\Phi(\mu_t^N) - \Phi(\mathcal{L}(X_t))\right|\right],$$

where $\Phi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is a test function. Typically $\Phi$ is linear (Bencheikh-Jourdain 2019), polynomial (Mischler-Mouhot 2013, Mischler-Mouhot-Wennberg 2015). Those approaches give a rate $O\left(\frac{1}{N}\right)$ which are not uniform in time. See also Chaudru de Raynal-Frikha (2021).

## Propagation of chaos III: weak errors

Weak errors are about the statistical behavior of the empirical distribution. Here, the focus is on deriving rates of convergence (in $t$ and $N$) for

$$\mathbb{E}\Big[\Big|\Phi(\mu_t^N) - \Phi(\mathcal{L}(X_t))\Big|\Big],$$

where $\Phi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is a test function. Typically $\Phi$ is linear (Bencheikh-Jourdain 2019), polynomial (Mischler-Mouhot 2013, Mischler-Mouhot-Wennberg 2015). Those approaches give a rate $O\left(\frac{1}{N}\right)$ which are not uniform in time. See also Chaudru de Raynal-Frikha (2021).

For the torus case, recent results of Delarue-Tse (2021) give, under regularity assumptions on $b$ and $\Phi$, the existence of a constant $C > 0$ such that for all $\mu_0 \in \mathcal{P}(\mathbb{T}^d)$,

$$\sup_{t\geq 0} \mathbb{E}\Big[\Big|\Phi(\mu_t^N) - \Phi(\mathcal{L}(X_t))\Big|\Big] \leq \frac{C}{N}.$$

# Beyond the mean-field limit for the MV particle system

# Controlling the correlations

As we have seen, we expect the contribution of $G_N^2$ to be of order $O\left(\frac{1}{N}\right)$. Keeping this contribution would allow to compute the correction to this mean-field limit, provided that we can also prove that $G_N^3$ is of order $O\left(\frac{1}{N^2}\right)$. And so on...

## Controlling the correlations

As we have seen, we expect the contribution of $G_N^2$ to be of order $O\left(\frac{1}{N}\right)$. Keeping this contribution would allow to compute the correction to this mean-field limit, provided that we can also prove that $G_N^3$ is of order $O\left(\frac{1}{N^2}\right)$. And so on...

Today's talk is about proving rigorously, the expected order of many-particles correlation in our smooth setting, i.e. showing that, in some weak sense and uniformly in time

$$G_N^{m+1} = O\left(\frac{1}{N^m}\right)$$

for all $m \geq 1$. Due to the complexity of those computations I will only present the case $m \in \{1, 2\}$ (but we have a clear road-map on how to go further).

## A brief reminder on cumulants

Our object of focus will be the cumulants (with respect to the different forms of randomness). Recall that cumulants of $(Z_1, \ldots, Z_n)$ measure, very roughly, the interactions between the variables. Typically they are useless is the random variables $Z_1, \ldots, Z_n$ are independent. We write $\kappa^m$ for the $m$-th global cumulant.

# A brief reminder on cumulants

Our object of focus will be the cumulants (with respect to the different forms of randomness). Recall that cumulants of $(Z_1, \dots, Z_n)$ measure, very roughly, the interactions between the variables. Typically they are useless is the random variables $Z_1, \dots, Z_n$ are independent. We write $\kappa^m$ for the $m$-th global cumulant.

Recall in particular that for all $X \in L^4(\Omega)$,

$$\kappa^2(X) = \mathrm{Var}(X), \qquad \kappa^3(X) = \mathbb{E}\Big[(X - \mathbb{E}[X])^3\Big].$$

But of course it is not always that easy

$$\kappa^4(X) = \mathbb{E}\Big[(X - \mathbb{E}[X])^4\Big] - 3\mathrm{Var}(X)^2.$$

# Main result

Our main result is the following:

## Theorem (B.-Duerinckx 2022$^+$)

*Assume that $b$ is given by a smooth, $H$-stable potential $W$, and that $\varphi : \mathbb{T}^d \to \mathbb{R}$ is any smooth function. Set $\Phi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ such that*

$$\Phi(\mu) = \int_{\mathbb{T}^d} \varphi(x)\mu(\mathrm{d}x).$$

*Then, for all $m \geq 1$, there exists a constant $C > 0$ such that, for any $\mu_\circ \in \mathcal{P}(\mathbb{T}^d)$,*

$$\sup_{t\geq 0} \kappa^{m+1}\left[\Phi(\mu_t^N)\right] \leq \frac{C}{N^m}.$$

# Main result

Our main result is the following:

## Theorem (B.-Duerinckx 2022$^+$)

*Assume that $b$ is given by a smooth, $H$-stable potential $W$, and that $\varphi : \mathbb{T}^d \to \mathbb{R}$ is any smooth function. Set $\Phi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ such that*

$$\Phi(\mu) = \int_{\mathbb{T}^d} \varphi(x)\mu(\mathrm{d}x).$$

*Then, for all $m \geq 1$, there exists a constant $C > 0$ such that, for any $\mu_\circ \in \mathcal{P}(\mathbb{T}^d)$,*

$$\sup_{t \geq 0} \kappa^{m+1}\left[\Phi(\mu_t^N)\right] \leq \frac{C}{N^m}.$$

Also, we expect an explicit dependency of $C$ in the derivatives of $\Phi$.

# Beyond the mean-field limit for the MV particle system

## The sources of randomness

We are talking about correlations... but where is the randomness in the system originating from ? There are two sources, and we will tackle them separately.

- The Brownian motions.
- The initial distributions.

Let us write $\mathbb{E}$ for the global randomness, $\mathbb{E}_\circ$ for the one related to the initial data, $\mathbb{E}_B$ for the one related to the Brownian motions. And so on, we write Var, $\mathrm{Var}_\circ$, $\mathrm{Var}_B$, $\kappa$, $\kappa_\circ$, $\kappa_B$...

# The sources of randomness

We are talking about correlations... but where is the randomness in the system originating from ? There are two sources, and we will tackle them separately.

- The Brownian motions.
- The initial distributions.

Let us write $\mathbb{E}$ for the global randomness, $\mathbb{E}_\circ$ for the one related to the initial data, $\mathbb{E}_B$ for the one related to the Brownian motions. And so on, we write Var, $\mathrm{Var}_\circ$, $\mathrm{Var}_B$, $\kappa$, $\kappa_\circ$, $\kappa_B$...

Recall that $\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{Y_t^{i,N}}$ for all $t \geq 0$. Consider a test function $\varphi : \mathbb{T}^d \to \mathbb{R}$, smooth, and set $\Phi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ defined by $\Phi(\mu) = \int_{\mathbb{T}^d} \varphi(x)\mu(\mathrm{d}x)$. For the second cumulant $G_N^2$, our goal is thus to show

$$\mathrm{Var}\left[\Phi(\mu_t^N)\right] = O\left(\frac{1}{N}\right),$$

uniformly in time.

## An illuminating example: the second-order cumulant

Let $\Phi$ be as before. We want to show

$$\mathrm{Var}\left[\Phi(\mu_t^N)\right] = O\left(\frac{1}{N}\right).$$

We split on the two sources of noise. I.e., we have

$$\mathrm{Var}\left[\Phi(\mu_t^N)\right] = \mathrm{Var}_\circ\left[\mathbb{E}_B[\Phi(\mu_t^N)]\right] + \mathbb{E}_\circ\left[\mathrm{Var}_B(\Phi(\mu_t^N))\right].$$

We will prove that $\mathbb{E}_B[\Phi(\mu_t^N)] = \Phi(m(t,\mu_0^N)) + O\left(\frac{1}{N}\right)$. For the second term, we will have to show that $\mathrm{Var}_B(\Phi(\mu_t^N)) = O\left(\frac{1}{N}\right)$.

## An illuminating example: the second-order cumulant

Let $\Phi$ be as before. We want to show

$$\mathrm{Var}\Big[\Phi(\mu_t^N)\Big] = O\Big(\frac{1}{N}\Big).$$

We split on the two sources of noise. I.e., we have

$$\mathrm{Var}\Big[\Phi(\mu_t^N)\Big] = \mathrm{Var}_\circ\Big[\mathbb{E}_B[\Phi(\mu_t^N)]\Big] + \mathbb{E}_\circ\Big[\mathrm{Var}_B(\Phi(\mu_t^N))\Big].$$

We will prove that $\mathbb{E}_B[\Phi(\mu_t^N)] = \Phi(m(t,\mu_0^N)) + O\Big(\frac{1}{N}\Big)$. For the second term, we will have to show that $\mathrm{Var}_B(\Phi(\mu_t^N)) = O\Big(\frac{1}{N}\Big)$.

This will be our guiding example throughout the talk.

# Our tools

What we see in particular is that the global cumulant can be split into cumulants with respect to the initial data of cumulants with respect to the Brownian motions. This is the case at all orders (law of joint cumulants).

# Our tools

What we see in particular is that the global cumulant can be split into cumulants with respect to the initial data of cumulants with respect to the Brownian motions. This is the case at all orders (law of joint cumulants).

We use:

- Expansions in the Wasserstein space and ergodic estimates to handle the Brownian cumulants. We were strongly inspired by recent works of Delarue-Tse (2021), Chassagneux-Szpruch-Tse (2019)...

# Our tools

What we see in particular is that the global cumulant can be split into cumulants with respect to the initial data of cumulants with respect to the Brownian motions. This is the case at all orders (law of joint cumulants).

We use:

- Expansions in the Wasserstein space and ergodic estimates to handle the Brownian cumulants. We were strongly inspired by recent works of Delarue-Tse (2021), Chassagneux-Szpruch-Tse (2019)...
- Glauber calculus to handle cumulants with respect to the initial distribution. Those were used by Duerinckx (2021) for the second-order system (with a velocity) without noise.

# Beyond the mean-field limit for the MV particle system

# Uniform in time control of the Brownian cumulants

In this part, we focus on the Brownian cumulants $\kappa_B^m(\Phi(\mu_t^N))$. Our goal is to prove that

## Proposition

*For all smooth $\varphi$ on $\mathbb{T}^d$, for $\Phi(\mu) = \int_{\mathbb{T}^d} \varphi(x)\mu(\mathrm{d}x)$, for all $m \geq 1$, there holds*

$$\kappa_B^{m+1}(\Phi(\mu_t^N)) = O\Big(\frac{1}{N^m}\Big),$$

*where the estimation is uniform in time.*

# Uniform in time control of the Brownian cumulants

In this part, we focus on the Brownian cumulants $\kappa_B^m(\Phi(\mu_t^N))$. Our goal is to prove that

## Proposition

*For all smooth $\varphi$ on $\mathbb{T}^d$, for $\Phi(\mu) = \int_{\mathbb{T}^d} \varphi(x)\mu(\mathrm{d}x)$, for all $m \geq 1$, there holds*

$$\kappa_B^{m+1}(\Phi(\mu_t^N)) = O\left(\frac{1}{N^m}\right),$$

*where the estimation is uniform in time.*

Coming back to the variance this is of course the key point to prove that $\mathrm{Var}_B[\Phi(\mu_t^N)] = O\left(\frac{1}{N}\right)$. Note however that this is not enough at all !For instance, if you consider the 4-th order global cumulant $\kappa^4$, we have terms of the form

$$\mathrm{Var}_\circ\left[\mathrm{Var}_B[\Phi(\mu_t^N)]\right]$$

# Beyond the mean-field limit for the MV particle system

# Linear functional derivatives

Let $F : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$. We say that $F$ is continuously differentiable if there exists a continuous function $\frac{\delta F}{\delta m} : \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d \to \mathbb{R}$ such that, for any $\mu, \mu' \in \mathcal{P}(\mathbb{T}^d)$,

$$F(\mu) - F(\mu') = \int_0^1 \int_{\mathbb{T}^d} \frac{\delta F}{\delta m}\big(s\mu + (1-s)\mu', y\big)(\mu - \mu')(\mathrm{d}y)\mathrm{d}s.$$

# Linear functional derivatives

Let $F : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$. We say that $F$ is continuously differentiable if there exists a continuous function $\frac{\delta F}{\delta m} : \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d \to \mathbb{R}$ such that, for any $\mu, \mu' \in \mathcal{P}(\mathbb{T}^d)$,

$$F(\mu) - F(\mu') = \int_0^1 \int_{\mathbb{T}^d} \frac{\delta F}{\delta m}\big(s\mu + (1-s)\mu', y\big)(\mu - \mu')(\mathrm{d}y)\mathrm{d}s.$$

The definition holds up to some additive constant, so we require

$$\int_{\mathbb{T}^d} \frac{\delta F}{\delta m}(\mu, y)\mu(\mathrm{d}y) = 0.$$

# Link with the Wasserstein (Otto) derivative

If we equip $\mathcal{P}(\mathbb{T}^d)$ with the Wasserstein distance $\mathcal{W}_1$, we have

$$F(\mu) - F(\mu') = \int_{\mathbb{T}^d} \frac{\delta F}{\delta m}(\mu', y)(\mu - \mu')(\mathrm{d}y) + o\Big(\mathcal{W}_1(\mu, \mu')\Big).$$

## Link with the Wasserstein (Otto) derivative

If we equip $\mathcal{P}(\mathbb{T}^d)$ with the Wasserstein distance $\mathcal{W}_1$, we have

$$F(\mu) - F(\mu') = \int_{\mathbb{T}^d} \frac{\delta F}{\delta m}(\mu', y)(\mu - \mu')(\mathrm{d}y) + o\Big(\mathcal{W}_1(\mu, \mu')\Big).$$

There is a related notion of Wasserstein derivative . In short, for $F : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$, the Wasserstein derivative $\partial_\mu F$ satisfies

$$\partial_\mu F(\nu, y) = \partial_y \frac{\delta F}{\delta m}(\nu, y).$$

When there is enough regularity, the derivatives commute. As we shall see, this notion is best suited for our framework. A clear dictionary between the two is established in Carmona-Delarue (2018).

# Higher-order derivatives

We can define higher-order linear derivatives: for $p \geq 2$, $y \in (\mathbb{T}^d)^{p-1}$,

$$\frac{\delta^{p-1} F}{\delta m^{p-1}}(\mu, y) - \frac{\delta^{p-1} F}{\delta m^{p-1}}(\mu', y) = \int_0^1 \int_{\mathbb{T}^d} \frac{\delta^p F}{\delta m^p}\big(s\mu + (1-s)\mu', y, y'\big)(\mu - \mu')(\mathrm{d}y')\mathrm{d}s$$

again with a condition $\int_{\mathbb{T}^d} \frac{\delta^p F}{\delta m^p}(\mu, y_1, \dots, y_p)\mu(\mathrm{d}y_p) = 0$ for $(y_1, \dots, y_{p-1}) \in (\mathbb{T}^d)^{p-1}$.

## Higher-order derivatives

We can define higher-order linear derivatives: for $p \geq 2$, $y \in (\mathbb{T}^d)^{p-1}$,

$$\frac{\delta^{p-1} F}{\delta m^{p-1}}(\mu, y) - \frac{\delta^{p-1} F}{\delta m^{p-1}}(\mu', y) = \int_0^1 \int_{\mathbb{T}^d} \frac{\delta^p F}{\delta m^p}\big(s\mu + (1-s)\mu', y, y'\big)(\mu - \mu')(\mathrm{d}y')\mathrm{d}s$$

again with a condition $\int_{\mathbb{T}^d} \frac{\delta^p F}{\delta m^p}(\mu, y_1, \ldots, y_p)\mu(\mathrm{d}y_p) = 0$ for $(y_1, \ldots, y_{p-1}) \in (\mathbb{T}^d)^{p-1}$.

Note that the $H$-stable framework from above provides the existence and the regularity of $\frac{\delta b}{\delta m}$ (and much more).

# The master equation

For any $\Phi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$, write $\mathcal{U}_\Phi(t,\mu) = \Phi(m(t,\mu))$ for $t \geq 0$, $\mu \in \mathcal{P}(\mathbb{T}^d)$. Then, from Buckdahn-Li-Peng-Rainer (2017), we have that $\mathcal{U}_\Phi$ satisfies the master equation

$$\begin{cases} \partial_t \mathcal{U}_\Phi(t,\mu) & = \int_{\mathbb{T}^d} \Big[ \sum_{i=1}^d \partial_{x_i} \frac{\delta \mathcal{U}_\Phi}{\delta m}(t,\mu,x) b_i(x,\mu) \\ & \qquad + \frac{1}{2} \sum_{i,j=1}^d \partial^2_{x_i x_j} \frac{\delta \mathcal{U}_\Phi}{\delta m}(t,\mu,x) \Big] \mu(\mathrm{d}x) \quad t \geq 0, \\ \mathcal{U}_\Phi(0,\mu) & = \Phi(\mu). \end{cases}$$

# The master equation

For any $\Phi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$, write $\mathcal{U}_\Phi(t,\mu) = \Phi(m(t,\mu))$ for $t \geq 0$, $\mu \in \mathcal{P}(\mathbb{T}^d)$. Then, from Buckdahn-Li-Peng-Rainer (2017), we have that $\mathcal{U}_\Phi$ satisfies the master equation

$$\begin{cases} \partial_t \mathcal{U}_\Phi(t,\mu) & = \int_{\mathbb{T}^d} \Big[ \sum_{i=1}^d \partial_{x_i} \frac{\delta \mathcal{U}_\Phi}{\delta m}(t,\mu,x) b_i(x,\mu) \\ & \qquad + \frac{1}{2} \sum_{i,j=1}^d \partial^2_{x_i x_j} \frac{\delta \mathcal{U}_\Phi}{\delta m}(t,\mu,x) \Big] \mu(\mathrm{d}x) \quad t \geq 0, \\ \mathcal{U}_\Phi(0,\mu) & = \Phi(\mu). \end{cases}$$

This result allows to expand functions of $m(t,\mu)$ along the dynamics. From Chassagneux-Szpruch-Tse (2019), we have

$$\mathbb{E}_B[\Phi(\mu_t^N)] = \mathcal{U}_\Phi(t,\mu_0^N) + \frac{1}{2N} \int_0^t \int_{\mathbb{T}^d} \mathbb{E}_B \Big[ \mathrm{Tr}[\partial^2_\mu \mathcal{U}_\Phi(t-s,\mu_s^N,v,v) \Big] \mu_s^N(\mathrm{d}v)\mathrm{d}s,$$

where we recall that $\partial_\mu \mathcal{U}_\Phi(t-s,\mu,y) = \partial_y \frac{\delta \mathcal{U}}{\delta m}(t-s,\mu,y)$.

## Pushing the expansion further

Set, for $0 \leq s \leq t$, $\mu \in \mathcal{P}(\mathbb{T}^d)$,

$$\Phi^{(1)}\big((t,s),\mu\big) = \int_{\mathbb{T}^d} \mathrm{Tr}\Big[\partial_\mu^2 \mathcal{U}_\Phi\big(t-s,\mu,y,y\big)\Big]\mu(\mathrm{d}y),$$

and then look at the flow corresponding to $\Phi^{(1)}$, i.e. set, for $0 \leq u \leq s \leq t$,

$$\mathcal{U}_\Phi^{(1)}\big((t,s,u),\mu\big) = \Phi^{(1)}\big((t,s),m(s-u,\mu)\big).$$

Then we can use $\mathcal{U}_\Phi^{(1)}$ to push the expansion further

## Pushing the expansion further

Set, for $0 \leq s \leq t$, $\mu \in \mathcal{P}(\mathbb{T}^d)$,

$$\Phi^{(1)}\big((t,s),\mu\big) = \int_{\mathbb{T}^d} \mathrm{Tr}\Big[\partial_\mu^2 \mathcal{U}_\Phi\big(t-s,\mu,y,y\big)\Big]\mu(\mathrm{d}y),$$

and then look at the flow corresponding to $\Phi^{(1)}$, i.e. set, for $0 \leq u \leq s \leq t$,

$$\mathcal{U}_\Phi^{(1)}\big((t,s,u),\mu\big) = \Phi^{(1)}\big((t,s), m(s-u,\mu)\big).$$

Then we can use $\mathcal{U}_\Phi^{(1)}$ to push the expansion further

$$\mathbb{E}_B[\Phi(\mu_t^N)] = \mathcal{U}_\Phi(t,\mu_0^N) + \frac{1}{2N}\int_0^t \mathcal{U}_\Phi^{(1)}\big((t,s,0),\mu_0^N\big)\mathrm{d}s$$
$$+ \frac{1}{4N^2}\int_0^t\int_0^s \mathrm{Tr}\Big[\partial_\mu^2 \mathcal{U}_\Phi^{(1)}\Big(\big(t,s,u\big),\mu_u^N,y,y\Big)\Big]\mu_u^N(\mathrm{d}y).$$

# Beyond the mean-field limit for the MV particle system

# Exploiting the decomposition → ergodic estimates

The key things is that we can now obtain our uniform in time controls through the linear derivatives appearing in the expansion. In fact, we can first represent those derivatives through linearized equations and have ergodic Sobolev estimates for those.

# Exploiting the decomposition → ergodic estimates

The key things is that we can now obtain our uniform in time controls through the linear derivatives appearing in the expansion. In fact, we can first represent those derivatives through linearized equations and have ergodic Sobolev estimates for those.

For instance,

$$\partial^2_{(x_1)_i(x_2)_j} \frac{\delta^2 \mathcal{U}_\Phi}{\delta m^2}(t,\mu,x_1,x_2) = -\varphi\big(d_j^{(1)}(t,\mu,z_2)\big) + \varphi\big(d_{i,j}^{(2)}(t,\mu,z_1,z_2)\big),$$

where $d_j^{(1)} \in \cap_{T>0} L^\infty([0,T],(W^{2,\infty}(\mathbb{T}^d))')$ solves

# Exploiting the decomposition → ergodic estimates

The key things is that we can now obtain our uniform in time controls through the linear derivatives appearing in the expansion. In fact, we can first represent those derivatives through linearized equations and have ergodic Sobolev estimates for those.

For instance,

$$\partial^2_{(x_1)_i(x_2)_j} \frac{\delta^2 \mathcal{U}_\Phi}{\delta m^2}(t,\mu,x_1,x_2) = -\varphi\big(d_j^{(1)}(t,\mu,z_2)\big) + \varphi\big(d_{i,j}^{(2)}(t,\mu,z_1,z_2)\big),$$

where $d_j^{(1)} \in \cap_{T>0} L^\infty([0,T],(W^{2,\infty}(\mathbb{T}^d))')$ solves

$$\partial_t d_j^{(1)} - \tfrac{1}{2}\triangle d_j^{(1)} + \operatorname{div}\Big(b(\cdot,m)d_j^{(1)}\Big) + \operatorname{div}\Big(m\frac{\delta b}{\delta m}(\cdot,m,d_j^{(1)})\Big) = 0,$$

and for $\xi \in W^{1,\infty}(\mathbb{T}^d)$, $\langle \xi, d_j^{(1)}(0,\mu,z)\rangle = \partial_{x_j}\xi(z)$. A similar equation is solved by $d^{(2)}$. Crucially, the form of our interaction allows to show that for any $\alpha > 0$, for some constant $C_1, C_2 > 0$,

$$\sup_{z\in\mathbb{T}^d}\ \sup_{\mu\in\mathcal{P}(\mathbb{T}^d)} \|d_i^{(1)}(t,\mu,z)\|_{(W^{1-\alpha,\infty})'} \le \frac{C_1}{1\wedge t^{\frac{\alpha}{2}}} e^{-C_2 t}.$$

## More on ergodic estimates

It seems that we can play a similar game for all the derivatives appearing in the expansion.
Those ergodic estimates rely very heavily on our hypothesis that the interaction is obtained through a $H$-stable potential. In particular, the knowledge of the spectrum is important.

# Beyond the mean-field limit for the MV particle system

## Back to the global variance

Using the ergodic estimates and the expansion, one can easily show that

$$\mathbb{E}_B[\Phi(\mu_t^N)] = \mathcal{U}_\Phi(t,\mu_0^N) + O\Big(\frac{1}{N}\Big), \tag{1}$$

as announced. For the variance, we write

$$\mathrm{Var}_B[\Phi(\mu_t^N)] = \mathbb{E}_B\Big[\Phi(\mu_t^N)^2\Big] - \mathbb{E}_B\Big[\Phi(\mu_t^N)\Big]^2,$$

and we look at the expansion in $\Phi$ and in $\Phi^2$.

## Back to the global variance

Using the ergodic estimates and the expansion, one can easily show that

$$\mathbb{E}_B[\Phi(\mu_t^N)] = \mathcal{U}_\Phi(t, \mu_0^N) + O\Big(\frac{1}{N}\Big), \tag{1}$$

as announced. For the variance, we write

$$\mathrm{Var}_B[\Phi(\mu_t^N)] = \mathbb{E}_B\Big[\Phi(\mu_t^N)^2\Big] - \mathbb{E}_B\Big[\Phi(\mu_t^N)\Big]^2,$$

and we look at the expansion in $\Phi$ and in $\Phi^2$.

First,

$$\mathbb{E}_B\Big[\Phi(\mu_t^N)^2\Big] = \mathcal{U}_{\Phi^2}(t, \mu_0^N) + O\Big(\frac{1}{N}\Big),$$

while it follows from (1) that the only first-order term of $\mathbb{E}_B[\Phi(\mu_t^N)]^2$ is $\mathcal{U}_\Phi(t, \mu_0^N)^2$. Moreover,

$$\mathcal{U}_{\Phi^2}(t, \mu_0^N) = \Phi^2(m(t, \mu_0^N)) = \mathcal{U}_\Phi(t, \mu_0^N)^2,$$

and there are no first-order terms, as expected.

## Back to the global variance

Using the ergodic estimates and the expansion, one can easily show that

$$\mathbb{E}_B[\Phi(\mu_t^N)] = \mathcal{U}_\Phi(t, \mu_0^N) + O\Big(\frac{1}{N}\Big), \tag{1}$$

as announced. For the variance, we write

$$\mathrm{Var}_B[\Phi(\mu_t^N)] = \mathbb{E}_B\Big[\Phi(\mu_t^N)^2\Big] - \mathbb{E}_B\Big[\Phi(\mu_t^N)\Big]^2,$$

and we look at the expansion in $\Phi$ and in $\Phi^2$.

First,

$$\mathbb{E}_B\Big[\Phi(\mu_t^N)^2\Big] = \mathcal{U}_{\Phi^2}(t, \mu_0^N) + O\Big(\frac{1}{N}\Big),$$

while it follows from (1) that the only first-order term of $\mathbb{E}_B[\Phi(\mu_t^N)]^2$ is $\mathcal{U}_\Phi(t, \mu_0^N)^2$. Moreover,

$$\mathcal{U}_{\Phi^2}(t, \mu_0^N) = \Phi^2(m(t, \mu_0^N)) = \mathcal{U}_\Phi(t, \mu_0^N)^2,$$

and there are no first-order terms, as expected.

We can play a similar (but really more involved) game for further cumulants.

# Beyond the mean-field limit for the MV particle system

## Back to the variance

Recall that

$$\mathrm{Var}[\Phi(\mu_t^N)] = \mathrm{Var}_\circ[\mathbb{E}_B[\Phi(\mu_t^N)]] + \mathbb{E}_\circ[\mathrm{Var}_B(\mu_t^N)].$$

We already know that $\mathrm{Var}_B(\mu_t^N) = O\left(\frac{1}{N}\right)$. Moreover, we have

$$\mathbb{E}_B\left[\Phi(\mu_t^N)\right] = \mathcal{U}_\Phi(t, \mu_0^N) + O\left(\frac{1}{N}\right).$$

How can we prove that $\mathrm{Var}_\circ\left[\mathbb{E}_B[\Phi(\mu_t^N)]\right] = O\left(\frac{1}{N}\right)$ ?

## Glauber derivatives

The idea is to use Glauber calculus as introduced by Duerinckx (2021) for the study of the second-order system without noise.

# Glauber derivatives

The idea is to use Glauber calculus as introduced by Duerinckx (2021) for the study of the second-order system without noise.

Define the Glauber derivative with respect to the $k$-th initial data $Y_0^k$ as

$$D_k^\circ Z = Z((Y_0^{1,N}, \dots, Y_0^{N,N})) - \int_{\mathbb{T}^d} Z(Y_0^{1,N}, \dots, z, \dots, Y_0^{N,N}) \mu_0(\mathrm{d}z).$$

Now, let again $\mathcal{U}_\Phi(t, \mu) = \Phi(m(t, \mu))$. We obtain

$$D_k^\circ \mathcal{U}_\Phi(t, \mu_0^N) = \mathcal{U}_\Phi(t, \mu_0^N) - \int_{\mathbb{T}^d} \mathcal{U}_\Phi\Big(t, \frac{1}{N} \sum_{j=1, j\neq k}^{N} \delta_{Y_0^{j,N}} + \frac{1}{N}\delta_z\Big)\mu_0(\mathrm{d}z).$$

## Glauber derivatives

The idea is to use Glauber calculus as introduced by Duerinckx (2021) for the study of the second-order system without noise.

Define the Glauber derivative with respect to the $k$-th initial data $Y_0^k$ as

$$D_k^\circ Z = Z((Y_0^{1,N},\dots,Y_0^{N,N})) - \int_{\mathbb{T}^d} Z(Y_0^{1,N},\dots,z,\dots,Y_0^{N,N})\mu_0(\mathrm{d}z).$$

Now, let again $\mathcal{U}_\Phi(t,\mu) = \Phi(m(t,\mu))$. We obtain

$$D_k^\circ \mathcal{U}_\Phi(t,\mu_0^N) = \mathcal{U}_\Phi(t,\mu_0^N) - \int_{\mathbb{T}^d} \mathcal{U}_\Phi\Big(t, \frac{1}{N}\sum_{j=1,j\neq k}^N \delta_{Y_0^{j,N}} + \frac{1}{N}\delta_z\Big)\mu_0(\mathrm{d}z).$$

Now, we can use the notion of linear derivatives ! Since $\int_{\mathbb{T}^d}\mu_0(\mathrm{d}z) = 1$, we get

$$D_k^\circ \mathcal{U}_\Phi(t,\mu_0^N) = \int_0^1 \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \frac{\delta \mathcal{U}_\Phi}{\delta m}\Big(t, \frac{1}{N}\sum_{j=1,j\neq k}^N \delta_{Y_0^{j,N}} + \frac{s}{N}\delta_{Y_0^{k,N}} + \frac{1-s}{N}\delta_z, y\Big)$$
$$\Big(\frac{1}{N}\delta_{Y_0^{k,N}} - \frac{1}{N}\delta_z\Big)(\mathrm{d}y)\mu_0(\mathrm{d}z)ds$$

## More on Glauber derivatives

Our ergodic estimates then allows to conclude that

$$D_k^\circ \mathbb{E}_B[\Phi(\mu_t^N)] = O\Big(\frac{1}{N}\Big).$$

To relate this to the variance, we use the Efron-Stein inequality

$$\mathrm{Var}_\circ[Y] \leq \mathbb{E}_\circ\Big[\sum_{j=1}^N |D_j^\circ Y|^2\Big].$$

This conclude our proof for the variance. For the higher-order terms, similar formula always allow to come back to iteration of those Glauber derivatives.

# Beyond the mean-field limit for the MV particle system

## The third cumulant

Let $\varphi$ smooth on $\mathbb{T}^d$, $\Phi(\mu) = \int_{\mathbb{T}^d} \varphi(x)\mu(\mathrm{d}x)$. We want to combine all of the previous arguments to obtain

$$\kappa^3\Big[\Phi(\mu_t^N)\Big] = O\Big(\frac{1}{N^2}\Big).$$

Note that, by the law of total cumulants

$$\kappa^3(X) = \kappa_\circ^3\Big(\mathbb{E}_B[X]\Big) + \mathbb{E}_\circ\Big(\kappa_B^3(X)\Big) + 3\mathrm{Cov}\Big(\mathrm{Var}_B(X), \mathbb{E}_B(X)\Big).$$

# The controls we will need

According to this decomposition we will need

1. A control of the third Brownian cumulant $\kappa_B^3[\Phi(\mu_t^N)] = O\left(\frac{1}{N^2}\right)$.

# The controls we will need

According to this decomposition we will need

1. A control of the third Brownian cumulant $\kappa_B^3[\Phi(\mu_t^N)] = O\left(\frac{1}{N^2}\right)$.
2. A control of the second Brownian moment with the exact value of the leading $O\left(\frac{1}{N}\right)$ term.

# The controls we will need

According to this decomposition we will need

1. A control of the third Brownian cumulant $\kappa_B^3[\Phi(\mu_t^N)] = O\left(\frac{1}{N^2}\right)$.
2. A control of the second Brownian moment with the exact value of the leading $O\left(\frac{1}{N}\right)$ term.
3. The two leading terms (of order 1 and $O\left(\frac{1}{N}\right)$) of the Brownian expectation.

We will then apply Glauber calculus for the Brownian variance and the Brownian expectation.

# Beyond the mean-field limit for the MV particle system

## Expectation and variance

By pushing to the second-order the expansion of Chassagneux-Tse and using the ergodic estimates, we immediately get

$$\mathbb{E}_B\left[\Phi(\mu_t^N)\right] = \mathcal{U}_\Phi(t,\mu_0^N) + \frac{1}{2N}\int_0^t \mathcal{U}_\Phi^{(1)}((t,s,0),\mu_0^N)ds + O\Big(\frac{1}{N^2}\Big).$$

## Expectation and variance

By pushing to the second-order the expansion of Chassagneux-Tse and using the ergodic estimates, we immediately get

$$\mathbb{E}_B\left[\Phi(\mu_t^N)\right] = \mathcal{U}_\Phi(t,\mu_0^N) + \frac{1}{2N}\int_0^t \mathcal{U}_\Phi^{(1)}((t,s,0),\mu_0^N)ds + O\Big(\frac{1}{N^2}\Big).$$

For the variance, we compare this with

$$\mathbb{E}_B\left[\Phi(\mu_t^N)^2\right] = \mathcal{U}_\Phi(t,\mu_0^N)^2 + \frac{1}{2N}\int_0^t \mathcal{U}_{\Phi^2}^{(1)}((t,s,0),\mu_0^N)\mathrm{d}s + O\Big(\frac{1}{N^2}\Big).$$

As we have seen, the two terms of order 1 cancel.

## Expectation and variance

By pushing to the second-order the expansion of Chassagneux-Tse and using the ergodic estimates, we immediately get

$$\mathbb{E}_B\left[\Phi(\mu_t^N)\right] = \mathcal{U}_\Phi(t,\mu_0^N) + \frac{1}{2N}\int_0^t \mathcal{U}_\Phi^{(1)}((t,s,0),\mu_0^N)ds + O\Big(\frac{1}{N^2}\Big).$$

For the variance, we compare this with

$$\mathbb{E}_B\left[\Phi(\mu_t^N)^2\right] = \mathcal{U}_\Phi(t,\mu_0^N)^2 + \frac{1}{2N}\int_0^t \mathcal{U}_{\Phi^2}^{(1)}((t,s,0),\mu_0^N)\mathrm{d}s + O\Big(\frac{1}{N^2}\Big).$$

As we have seen, the two terms of order 1 cancel.

For the leading order term, we obtain

$$\begin{aligned}\mathrm{Var}_B\left[\Phi(\mu_t^N)\right] = \frac{1}{2N}\int_0^t \mathcal{U}_{\Phi^2}^{(1)}((t,s,0),\mu_0^N)ds - \frac{\mathcal{U}_\Phi(t,\mu_0^N)}{N}\int_0^t \mathcal{U}_\Phi^{(1)}((t,s,0),\mu_0^N)ds\\ + O\Big(\frac{1}{N^2}\Big).\end{aligned}$$

# Brownian variance II

Now, since $\mathcal{U}_{\Phi^2} = (\mathcal{U}_\Phi)^2$, and

$$\partial_\mu^2(g^2)(x,x) = 2g\partial_\mu^2 g(x,x) + 2(\partial_\mu g(x))^2$$

the first term on the rhs splits thanks to the fact that for $0 \le u \le s \le t$, $\mu \in \mathcal{P}(\mathbb{T}^d)$,

$$\begin{aligned}\mathcal{U}^{(1)}_{\Phi^2}((t,s,u),\mu) &= \int_{\mathbb{T}^d} \mathrm{Tr}\Big[\partial_\mu^2 \big(\mathcal{U}_\Phi\big)^2 (t-s, m(s-u,\mu), y, y)\Big] m(s-u,\mu)(\mathrm{d}y) \\ &= 2\mathcal{U}_\Phi\big(t-s, m(s-u,\mu)\big) \int_{\mathbb{T}^d} \mathrm{Tr}\Big[\partial_\mu^2 \mathcal{U}_\Phi\big(t-s, m(s-u,\mu), y, y\big)\Big] m(s-u,\mu)(\mathrm{d}y) \\ &\quad + 2\int_{\mathbb{T}^d} \Big|\partial_\mu \mathcal{U}_\Phi\big(t-s, m(s-u,\mu), y\big)\Big|^2 m(s-u,\mu)(\mathrm{d}y).\end{aligned}$$

# Brownian variance II

Now, since $\mathcal{U}_{\Phi^2} = (\mathcal{U}_\Phi)^2$, and

$$\partial_\mu^2 (g^2)(x,x) = 2g\partial_\mu^2 g(x,x) + 2(\partial_\mu g(x))^2$$

the first term on the rhs splits thanks to the fact that for $0 \le u \le s \le t$, $\mu \in \mathcal{P}(\mathbb{T}^d)$,

$$\begin{aligned}
\mathcal{U}^{(1)}_{\Phi^2}((t,s,u),\mu) &= \int_{\mathbb{T}^d} \mathrm{Tr}\Big[\partial_\mu^2 (\mathcal{U}_\Phi)^2 (t-s, m(s-u,\mu), y, y)\Big] m(s-u,\mu)(\mathrm{d}y) \\
&= 2\mathcal{U}_\Phi\left(t-s, m(s-u,\mu)\right) \int_{\mathbb{T}^d} \mathrm{Tr}\Big[\partial_\mu^2 \mathcal{U}_\Phi\left(t-s, m(s-u,\mu), y, y\right)\Big] m(s-u,\mu)(\mathrm{d}y) \\
&\quad + 2\int_{\mathbb{T}^d} \Big|\partial_\mu \mathcal{U}_\Phi\left(t-s, m(s-u,\mu), y\right)\Big|^2 m(s-u,\mu)(\mathrm{d}y).
\end{aligned}$$

Note that, by the flow property:

$$\mathcal{U}_\Phi\left(t-s, m(s,\mu_0^N)\right) = \Phi\Big(m(t-s, m(s,\mu_0^N))\Big) = \Phi(m(t,\mu_0^N)) = \mathcal{U}_\Phi(t,\mu_0^N),$$

so that the first term in the previous decomposition (with $u=0, \mu = \mu_0^N$) is exactly the crossed term from $\mathbb{E}_B[\Phi(\mu_t^N)]^2$ !

# Brownian variance II

Now, since $\mathcal{U}_{\Phi^2} = (\mathcal{U}_\Phi)^2$, and

$$\partial^2_\mu (g^2)(x,x) = 2g\partial^2_\mu g(x,x) + 2(\partial_\mu g(x))^2$$

the first term on the rhs splits thanks to the fact that for $0 \le u \le s \le t$, $\mu \in \mathcal{P}(\mathbb{T}^d)$,

$$\begin{aligned}
\mathcal{U}^{(1)}_{\Phi^2}((t,s,u),\mu) &= \int_{\mathbb{T}^d} \mathrm{Tr}\Big[\partial^2_\mu (\mathcal{U}_\Phi)^2 (t-s, m(s-u,\mu), y, y)\Big] m(s-u,\mu)(\mathrm{d}y) \\
&= 2\mathcal{U}_\Phi\left(t-s, m(s-u,\mu)\right) \int_{\mathbb{T}^d} \mathrm{Tr}\Big[\partial^2_\mu \mathcal{U}_\Phi\left(t-s, m(s-u,\mu), y, y\right)\Big] m(s-u,\mu)(\mathrm{d}y) \\
&\quad + 2\int_{\mathbb{T}^d} \Big|\partial_\mu \mathcal{U}_\Phi\left(t-s, m(s-u,\mu), y\right)\Big|^2 m(s-u,\mu)(\mathrm{d}y).
\end{aligned}$$

Note that, by the flow property:

$$\mathcal{U}_\Phi\left(t-s, m(s,\mu^N_0)\right) = \Phi\Big(m(t-s, m(s,\mu^N_0))\Big) = \Phi(m(t,\mu^N_0)) = \mathcal{U}_\Phi(t,\mu^N_0),$$

so that the first term in the previous decomposition (with $u=0, \mu = \mu^N_0$) is exactly the crossed term from $\mathbb{E}_B[\Phi(\mu^N_t)]^2$ !
Hence

$$\mathrm{Var}_B\left[\Phi(\mu^N_t)\right] = \frac{1}{N}\int_0^t \Big|\partial_\mu \mathcal{U}_\Phi\left(t-s, m(s,\mu^N_0), y\right)\Big|^2 m(s,\mu^N_0)(\mathrm{d}y).$$

# Beyond the mean-field limit for the MV particle system

## Third Brownian cumulant

For the third Brownian cumulant, we also need to expand $\Phi^3$.

$$\mathbb{E}_B\left[\Phi^3(\mu_t^N)\right] = \mathcal{U}_\Phi(t,\mu_0^N)^3 + \frac{1}{2N}\int_0^t \mathcal{U}^{(1)}_{\Phi^3}((t,s),\mu_0^N)\mathrm{d}s + O\left(\frac{1}{N^2}\right).$$

Recall that $\kappa_B^3(X) = \mathbb{E}_B[X^3] - 3\mathbb{E}_B[X^2]\mathbb{E}_B[X] + 2\mathbb{E}_B[X]^3$. Clearly, comparing the 1-th order term gives 0.

## Third Brownian cumulant

For the third Brownian cumulant, we also need to expand $\Phi^3$.

$$\mathbb{E}_B\left[\Phi^3(\mu_t^N)\right] = \mathcal{U}_\Phi(t,\mu_0^N)^3 + \frac{1}{2N}\int_0^t \mathcal{U}^{(1)}_{\Phi^3}((t,s),\mu_0^N)\mathrm{d}s + O\Big(\frac{1}{N^2}\Big).$$

Recall that $\kappa_B^3(X) = \mathbb{E}_B[X^3] - 3\mathbb{E}_B[X^2]\mathbb{E}_B[X] + 2\mathbb{E}_B[X]^3$. Clearly, comparing the 1-th order term gives 0.

For the $\frac{1}{N}$-order terms, we get

$$\begin{aligned}&\frac{1}{2N}\int_0^t \mathcal{U}^{(1)}_{\Phi^3}((t,s),\mu_0^N)\mathrm{d}s - \frac{3}{2N}\mathcal{U}_\Phi(t,\mu_0^N)^2\int_0^t \mathcal{U}^{(1)}_{\Phi}((t,s),\mu_0^N)\mathrm{d}s\\ &\quad - \frac{3}{2N}\mathcal{U}_\Phi(t,\mu_0^N)\int_0^t \mathcal{U}^{(1)}_{\Phi^2}((t,s),\mu_0^N)\mathrm{d}s + \frac{3}{N}\mathcal{U}_\Phi(t,\mu_0^N)^2\int_0^t \mathcal{U}^{(1)}_{\Phi}((t,s),\mu_0^N)\mathrm{d}s.\end{aligned}$$

Moreover,

# Third Brownian cumulant

For the third Brownian cumulant, we also need to expand $\Phi^3$.

$$\mathbb{E}_B\left[\Phi^3(\mu_t^N)\right] = \mathcal{U}_\Phi(t,\mu_0^N)^3 + \frac{1}{2N}\int_0^t \mathcal{U}^{(1)}_{\Phi^3}((t,s),\mu_0^N)\mathrm{d}s + O\Big(\frac{1}{N^2}\Big).$$

Recall that $\kappa_B^3(X) = \mathbb{E}_B[X^3] - 3\mathbb{E}_B[X^2]\mathbb{E}_B[X] + 2\mathbb{E}_B[X]^3$. Clearly, comparing the 1-th order term gives 0.

For the $\frac{1}{N}$-order terms, we get

$$\begin{aligned}\frac{1}{2N}\int_0^t \mathcal{U}^{(1)}_{\Phi^3}((t,s),\mu_0^N)\mathrm{d}s - \frac{3}{2N}\mathcal{U}_\Phi(t,\mu_0^N)^2\int_0^t \mathcal{U}^{(1)}_{\Phi}((t,s),\mu_0^N)\mathrm{d}s \\ - \frac{3}{2N}\mathcal{U}_\Phi(t,\mu_0^N)\int_0^t \mathcal{U}^{(1)}_{\Phi^2}((t,s),\mu_0^N)\mathrm{d}s + \frac{3}{N}\mathcal{U}_\Phi(t,\mu_0^N)^2\int_0^t \mathcal{U}^{(1)}_{\Phi}((t,s),\mu_0^N)\mathrm{d}s.\end{aligned}$$

Moreover,

$$\begin{aligned}\partial_\mu^2[f^3](x,x) - 3f\partial_\mu^2[f^2](x,x) &= 6f(\partial_\mu f(x))^2 + 3f^2\partial_\mu^2 f(x,x) \\ &\quad - 6f^2\partial_\mu^2 f(x,x) - 6f(\partial_\mu f(x))^2\end{aligned}$$

so that ultimately, those terms cancel.

# Beyond the mean-field limit for the MV particle system

# Conclusion

Having identified each term, we can now apply the Glauber calculus from the previous section, and we recover the result thanks to the decomposition due to the law of joint cumulants.

# Conclusion

Having identified each term, we can now apply the Glauber calculus from the previous section, and we recover the result thanks to the decomposition due to the law of joint cumulants.

This gives a clear strategy to treat the cumulants at any order. The structure of the leading order terms for any $\kappa_B^m$ is not completely clear at this point, but we hope to be able to derive something systematic (similarly to the games on derivatives that we have just played).

# Perspectives

What remains to be done

- Tracking precisely the required regularity of $\varphi$ (and the one of $b$ ? What happens in a non-smooth case ?)

# Perspectives

What remains to be done

- Tracking precisely the required regularity of $\varphi$ (and the one of $b$ ? What happens in a non-smooth case ?)
- Adding a (convex) potential ? Going to the whole space ?

# Perspectives

What remains to be done

- Tracking precisely the required regularity of $\varphi$ (and the one of $b$ ? What happens in a non-smooth case ?)
- Adding a (convex) potential ? Going to the whole space ?
- Going to the second-order system. The transition should be easy except for the ergodic estimates, where things are not so clear. Very likely we will have to rely on linear hypocoercivity techniques (luckily I like kinetic theory).

# Perspectives

What remains to be done

- Tracking precisely the required regularity of $\varphi$ (and the one of $b$ ? What happens in a non-smooth case ?)
- Adding a (convex) potential ? Going to the whole space ?
- Going to the second-order system. The transition should be easy except for the ergodic estimates, where things are not so clear. Very likely we will have to rely on linear hypocoercivity techniques (luckily I like kinetic theory).
- Applications: What is the correction of the mean-field limit in the second-order case, when starting with uniform in space initial data ? Is it really $O\left(\frac{1}{N}\right)$ ?

# Perspectives

What remains to be done

- Tracking precisely the required regularity of $\varphi$ (and the one of $b$ ? What happens in a non-smooth case ?)
- Adding a (convex) potential ? Going to the whole space ?
- Going to the second-order system. The transition should be easy except for the ergodic estimates, where things are not so clear. Very likely we will have to rely on linear hypocoercivity techniques (luckily I like kinetic theory).
- Applications: What is the correction of the mean-field limit in the second-order case, when starting with uniform in space initial data ? Is it really $O\left(\frac{1}{N}\right)$ ?

Thank you for your attention !