Convergence towards the steady state of a collisionless gas with Cercignani-Lampis boundary condition.

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Problem and main result

We consider a C^2 bounded domain D in $\mathbb{R}^n$ for $n=3$, ∂D its boundary. We write n_x for the unit outward normal vector at $x\in\partial D$. Inspired by physical models [KS20] we study the kinetic free-transport equation

$$\partial_t f + v\cdot\nabla_x f = 0 \quad (x,v)\in D\times\mathbb{R}^n, \tag{1}$$

completed with a boundary condition. We consider the **Cercignani-Lampis** boundary condition, a generalisation of the diffuse boundary condition treated in [Ber20]. Setting $\partial_\pm G := \{(x,v)\in\partial D\times\mathbb{R}^n, \pm(v\cdot n_x)>0\}$, for all $(x,v)\in\partial_- G$, $t\geq 0$,

$$f(t,x,v) = \int_{\{v'\cdot n_x>0\}} f(t,x,v')|v'\cdot n_x| R(v'\to v;x)dv', \tag{2}$$

where $R(u\to v;x)$ is given, for $(x,u)\in\partial_+G$, $v\in\mathbb{R}^n$, with $v\cdot n_x<0$, by

$$R(u\to v;x) := \frac{1}{\theta(x)r_\perp}\frac{1}{(2\pi\theta(x)r_\parallel(2-r_\parallel))^{\frac{d-1}{2}}}\exp\Big(-\frac{|v_\perp|^2}{2\theta(x)r_\perp}-\frac{(1-r_\perp)|u_\perp|^2}{2\theta(x)r_\perp}\Big)$$
$$\times\exp\Big(-\frac{|v_\parallel-(1-r_\parallel)u_\parallel|^2}{2\theta(x)r_\parallel(2-r_\parallel)}\Big)I_0\Big(\frac{(1-r_\perp)^{\frac12}u_\perp\cdot v_\perp}{\theta(x)r_\perp}\Big), \tag{3}$$

where $v_\perp=(v\cdot n_x)n_x$, $v_\parallel = v - v_\perp$ and the corresponding definitions for $u_\perp, u_\parallel$. Here, I_0 is the modified Bessel function given, for all $y\in\mathbb{R}$, by

$$I_0(y) := \frac{1}{\pi}\int_0^\pi \exp\Big(y\cos\phi\Big)\,d\phi, \tag{4}$$

$\theta:\partial D\to(0,\infty)$ is the temperature at each point $x\in\partial D$, and we have two accommodation coefficients, $r_\perp\in(0,1)$ and $r_\parallel\in(0,2)$, for, respectively, the normal and the tangential velocities.

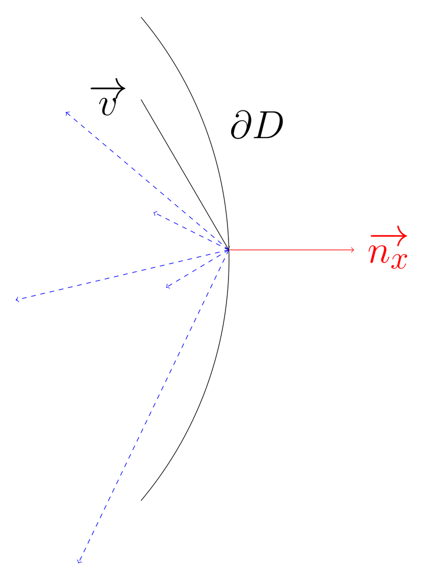

Figure 1: Diffuse reflection at the boundary, corresponding to $r_\perp=r_\parallel=1$. Possible outcoming velocities in blue.

This problem corresponds to the behavior of a collisionless gas (Knudsen) enclosed in a vessel. Starting from $f_0\in L^1(D\times\mathbb{R}^n)$, the solution to the problem (1), (2) such that

$$f(0,x,v) = f_0(x,v) \text{ a.e. in } D\times\mathbb{R}^n, \tag{5}$$

exists and is unique in the L^1 sense. We are interested here in the long-time behavior of this solution: does it converges towards some steady state ? If so, what is the rate of this convergence ? We prove that

1. there exists indeed a steady state towards which the solution converges in $L^1(D\times\mathbb{R}^n)$ as $t\to\infty$;
2. the rate of this convergence is $\frac{1}{(t+1)^{n-}}$ for any reasonable f_0. This extends the results known for the case $r_\parallel = r_\perp = 1$, and gives another good example of "weak (hypo)dissipativity".

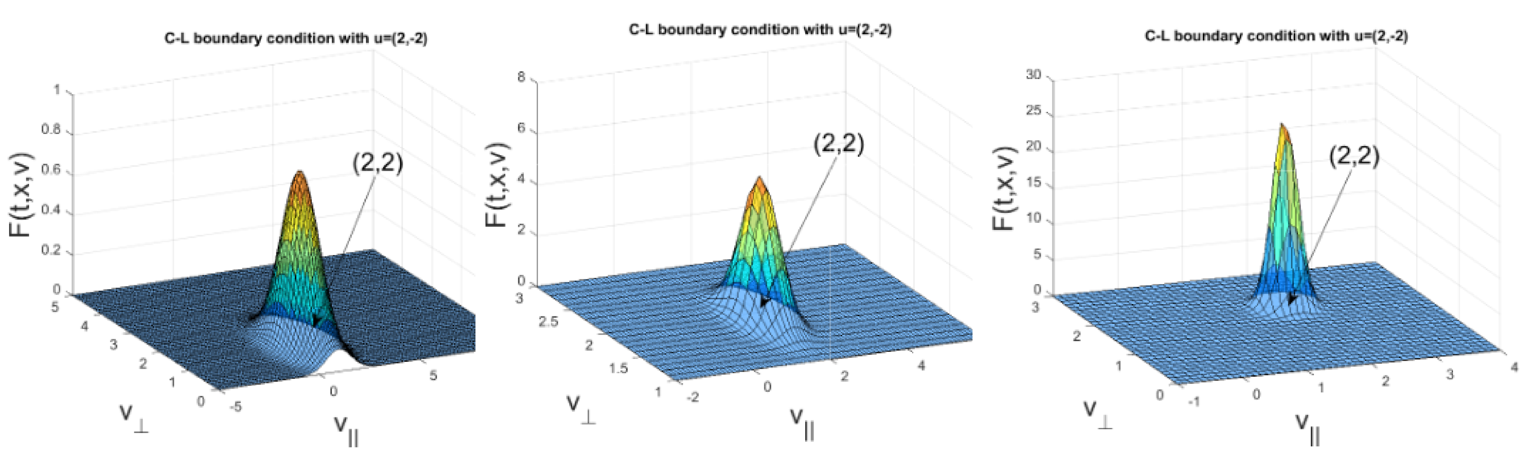

Figure 2: The Cercignani Lampis boundary condition in 2D with $\theta=1$ and $r_\perp=r_\parallel=0.5$ (left), $r_\perp=r_\parallel=0.1$ (center), $r_\perp=r_\parallel=1/30$ (right). Figures are taken from Chen [Che20].

Well-posedness, mass conservation and semigroup

The problem (1)-(2)-(5) satisfies positivity, a result due to the very simple form of characteristics and to the fact that the boundary operator is itself positive. The trace of an L^1 solution f to (1) exists at the boundary, we write it γf. The mass conservation is easily obtained using the normalization property of R: for all $(x,u)\in\partial_+G$,

$$\int_{\{v\cdot n_x<0\}} R(u\to v;x)\,|v\cdot n_x|\,dv = 1$$

We can thus associate to the problem a semigroup of operators $(S_t)_{t\geq0}$ satisfying mass conservation. Given $f_0\in L^1(D\times\mathbb{R}^n)$ an initial datum, $t\geq 0$ $S_tf_0 = f(t,x,v)$ is the solution to (1, 2, 5) at time t. From now on we consider the difference between two solutions, hence f_0 is of mass 0.

A Subgeometric Lyapunov Inequality

We introduce the function

$$\sigma(x,v) = \begin{cases}\inf\{s>0: x+sv\in\partial D\} \text{ if } (x,v)\in(D\times\mathbb{R}^n)\cup\partial_-G,\\ 0 \text{ otherwise }.\end{cases}$$

From [Voi80], this function satisfies $v\cdot\nabla_x\sigma(x,v) = -1$ a.e. in $D\times\mathbb{R}^n$, and behaves as $\frac{1}{|v|}$ when $|v|\to 0$. We consider, for $i\geq 0$, the weights $m_i = (1+\sigma(x,v)+\sqrt{|v|})^i$ and $\|g\|_{m_i} = \|gm_i\|_{L^1}$ for all $g\in L^1(D\times\mathbb{R}^n)$. One can show that

$$\|S_Tf\|_{m_i} + \kappa\int_0^T \|S_sf\|_{m_{i-1}}ds \leq \|f\|_{m_i} + b(1+T)\|f\|_{L^1}, \tag{6}$$

for $1\leq i\leq(n+1)-$, for all $T>0$ and with some constants $b,\kappa>0$. The upper bound on i comes from the necessary condition in the proof that for all $(x,u)\in\partial_+G$,

$$\int_{\{v\cdot n_x<0\}} m_i(x,v)R(u\to v;x)|v\cdot n_x|dv<\infty.$$

Deriving this Lyapunov inequality is the delicate part of the adaptation of the proof from the diffuse boundary condition. Indeed, the boundary operator is no longer weakly compact (we retain some information from the incoming velocity). On the other hand, if v is the incoming velocity, and v' the outcoming one, $|v'|\leq|v|$ on average.

Doeblin-Harris Condition

Using the nice forms of the characteristics for the problem (1)-(2)-(5), we can show that, for all $\rho>0$ there exists $T(\rho)>0$ satisfying for some measure $\nu\not\equiv 0$,

$$S_{T(\rho)}f \geq \nu\int_{\{m_1(x,v)\leq\rho\}} f\,dxdv, \qquad \forall f\in L^1(D\times\mathbb{R}^n), f\geq 0. \tag{7}$$

Sketch of proof in dimension 3

We combine (7) and (6) with the method tailored in [CM21]. We derive the following alternative:

$$\|f\|_{m_2}\leq A\|f\|_{L^1}, \quad\text{ or }\quad \|f\|_{m_2}>A\|f\|_{L^1},$$

for some well-chosen A. In both case, for $\|.\|_{\alpha,\beta} = \|.\|_{L^1}+\beta\|.\|_{m_{3-}}+\alpha\|f\|_{m_2}$, we prove that for some $\alpha>0,\beta>0$,

$$\|S_Tf\|_{\alpha,\beta}\leq\|f\|_{\alpha,\beta},$$

from which we conclude $\|S_Tf\|_{m_{3-}}\leq M_3\|f\|_{m_{3-}}$ for all $T>0$, some $M_3>0$.

Consider the norm $\|.\|_{\alpha,\beta,1} = \|.\|_{L^1}+\beta\|.\|_{m_1}+\alpha\|.\|_{m_0}$. With a similar computation, for some Z constant, using

$$m_1\leq\lambda m_0+\epsilon_\lambda m_{3-},$$

with $\epsilon_\lambda = \frac{1}{\lambda^{n-}}$, we obtain

$$Z\|S_Tf\|_{\alpha,\beta,1}\leq\|f\|_{\alpha,\beta,1}+\alpha\frac{\epsilon_\lambda}{\lambda}\|S_Tf\|_{m_{3-}}.$$

Iterating this result, we conclude that for all $t\geq0$,

$$\|S_tf\|_{\alpha,\beta,1}\lesssim\Big(\frac{1}{(t+1)^{(n-1)-}}\Big)\|f\|_{m_{3-}},$$

reinjecting this in

$$\|S_Tf\|_{\alpha,\beta,1}+2\alpha\|S_Tf\|_{m_0}\leq\|f\|_{\alpha,\beta,1},$$

and iterating, we gain one more exponent to conclude that for all $t>0$,

$$\|S_tf\|_{L^1}\lesssim\frac{1}{(t+1)^{n-}}\|f\|_{m_3-}.$$

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