Rate of convergence towards equilibrium for a collisionless gas

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Problem and main result

We consider a C^2 bounded domain D in $\mathbb{R}^n$ for $n \in \{2,3\}$, ∂D its boundary. We write n_x for the unit inward normal vector at $x \in \partial D$. As in [TAG10, AG11], we study the kinetic free-transport equation

$$\partial_t f + v \cdot \nabla_x f = 0 \quad (x,v) \in D \times \mathbb{R}^n, \tag{1}$$

completed with a boundary condition. We consider the so-called diffuse reflexion, for all $(x,v) \in \partial_+ G = \{(x,v) \in \partial D \times \mathbb{R}^n, v \cdot n_x > 0\}$,

$$f(t,x,v)(v \cdot n_x) = (v \cdot n_x) c M(v) \Big(\int_{v' \cdot n_x > 0} f(t,x,v') |v' \cdot n_x| dv' \Big), \tag{2}$$

with c a normalizing constant.

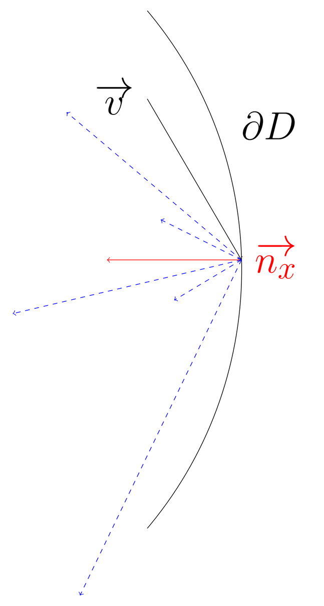

Figure 1: Diffuse reflection at the boundary. Possible outcoming velocities in blue.

The function M is a Gaussian distribution

$$M(v) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{\|v\|^2}{2}}.$$

This problem corresponds to the behavior of a collisionless gas (Knudsen) enclosed in a vessel. Starting from $f_0 \in L^1(D \times \mathbb{R}^n)$, the solution to the problem (1), (2) such that

$$f(0,x,v) = f_0(x,v) \text{ a.e. in } D \times \mathbb{R}^n, \tag{3}$$

exists, is unique, and converges in the L^1 sense towards an equilibrium given by

$$\mu_\infty(x,v) = \frac{M(v)}{|D|} \int_{D \times \mathbb{R}^n} f_0(x,v) dx dv.$$

We prove by semigroup arguments that the rate of this convergence is $\frac{1}{(t+1)^{n-}}$, more precisely $\frac{\ln(t+1)^{n+1}}{(t+1)^n}$ for any "reasonable" f_0. This model is a good example of "weak (hypo)dissipativity".

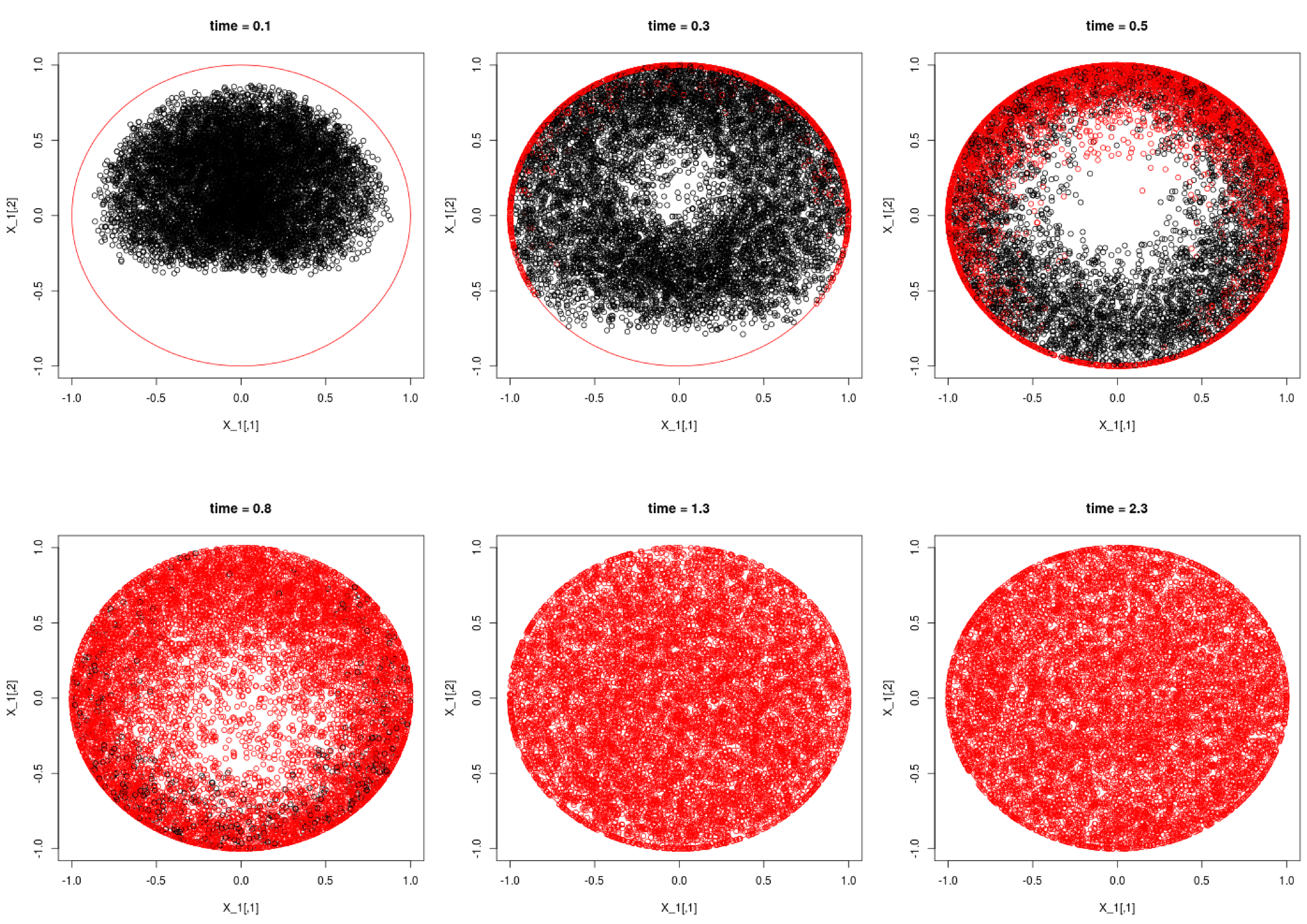

Figure 2: Convergence towards equilibrium. In red, particles which touched the boundary.

Well-posedness, mass conservation and semigroup

The problem (1, 2, 3) satisfies positivity, a result due to the very simple form of characteristics and to the fact that the boundary operator is itself positive. The trace of an L^1 solution f to (1) exists at the boundary, we write it γf. The mass conservation is easily obtained by Green's theorem:

$$\frac{d}{dt} \int_{D \times \mathbb{R}^n} f(t,x,v) dx dv = - \int_{D \times \mathbb{R}^n} v \cdot \nabla_x f(t,x,v) dx dv$$
$$= \int_{\partial D \times \mathbb{R}^n} \gamma f(t,x,v)(v \cdot n_x) dv dS(x),$$

with $dS(x)$ the surface measure of ∂D at x and where the sign is due to the fact that n_x is the inward normal vector at x. Since $c \int_{v \cdot n_x > 0} M(v) |v \cdot n_x| dv = 1$, we easily have

$$\int_{v \cdot n_x > 0} \gamma f |v \cdot n_x| dv = \int_{v \cdot n_x < 0} \gamma f |v \cdot n_x| dv,$$

from which we conclude. We can thus associate to the problem a semigroup of operators $(S_t)_{t \geq 0}$, such that

1. $S_0 = Id$, $S_{t+s} = S_t S_s$ for all $s \geq 0, t \geq 0$,
2. $S_t \geq 0$ for all $t \geq 0$,
3. $t \to S_t$ is (strongly) continuous, i.e. for all $f \in L^1(D \times \mathbb{R}^n)$, $\|S_t f - f\|_{L^1} \to 0$ as $t \to 0$,
4. $\int_{D \times \mathbb{R}^n} S_t f dx dv = \int_{D \times \mathbb{R}^n} f dx dv$ for all $t \geq 0$ (mass conservation).

Given $f_0 \in L^1(D \times \mathbb{R}^n)$ an initial datum, $t \geq 0$ $S_t f_0 = f(t,x,v)$ is the solution to (1, 2, 3) at time t. From now on we write f_0 for $f_0 - \mu_\infty$, so that f_0 is of mass 0.

A Subgeometric Lyapunov Inequality

We introduce the function

$$\sigma(x,v) = \begin{cases} \inf\{s > 0 : x + sv \in \partial D\} \text{ if } (x,v) \in (D \times \mathbb{R}^n) \cup \partial_+ G, \\ 0 \text{ otherwise }. \end{cases}$$

From [EGKM13], this function satisfies $v \cdot \nabla_x \sigma(x,v) = -1$ a.e. in $D \times \mathbb{R}^n$, and behaves as $\frac{1}{\|v\|}$ when $\|v\| \to 0$. We consider, for $i \geq 0$, the weights $m_i = (e^2 + \sigma(x,v))^i$ and $\|g\|_{m_i} = \|g m_i\|_{L^1}$ for all $g \in L^1(D \times \mathbb{R}^n)$. One can show that

$$\|S_T f\|_{m_i} - \kappa \int_0^T \|S_s f\|_{m_{i-1}} \leq \|f\|_{m_i} + b(1+T) \|f\|_{L^1}, \tag{4}$$

for $1 \leq i \leq (n+1)-$, for all $T > 0$ and with some constants $b, \kappa > 0$. The upper bound on i comes from the necessary condition in the proof that

$$\int_{v \cdot n_x > 0} m_i(x,v) M(v) |v \cdot n_x| dv < \infty.$$

Doeblin-Harris Condition

Using the nice forms of the characteristics for the problem (1, 2, 3), we can show that, for all $\rho > 0$ there exists $T(\rho) > 0$ satisfying for some measure $\nu \not\equiv 0$,

$$S_{T(\rho)} f \geq \nu \int_{\{\sigma \leq \rho\}} f dx dv, \qquad \forall f \in L^1(D \times \mathbb{R}^n), f \geq 0. \tag{5}$$

Sketch of proof in dimension 3

We conclude with the help of (5) and (4). We derive the following alternative:

$$\|f\|_{m_2} \leq A \|f\|_{L^1}, \quad \text{or} \quad \|f\|_{m_2} > A \|f\|_{L^1},$$

for some well-chosen A. In both case, for $\|.\|_{\alpha,\beta} = \|.\|_{L^1} + \beta \|.\|_{m_{3-}} + \alpha \|f\|_{m_2}$, we prove that for some $\alpha > 0, \beta > 0$,

$$\|S_T f\|_{\alpha,\beta} \leq \|f\|_{\alpha,\beta},$$

from which we conclude $\|S_T f\|_{m_{3-}} \leq M_3 \|f\|_{m_{3-}}$ for all $T > 0$, some $M_3 > 0$.

Consider the norm $\|.\|_{\alpha,\beta,1} = \|.\|_{L^1} + \beta \|.\|_{m_1} + \alpha \|.\|_{m_0}$. With a similar computation, for some Z constant, using

$$m_1 \leq \lambda m_0 + \epsilon_\lambda m_{3-},$$

with $\epsilon_\lambda = \frac{1}{\lambda^{n-}}$, we obtain

$$Z \|S_T f\|_{\alpha,\beta,1} \leq \|f\|_{\alpha,\beta,1} + \alpha \frac{\epsilon_\lambda}{\lambda} \|S_T f\|_{m_{3-}}.$$

Iterating this result, we conclude that for all $t \geq 0$,

$$\|S_t f\|_{\alpha,\beta,1} \lesssim \Big(\frac{1}{(t+1)^{(n-1)-}} \Big) \|f\|_{m_{3-}},$$

reinjecting this in

$$\|S_T f\|_{\alpha,\beta,1} + 2\alpha \|S_T f\|_{m_0} \leq \|f\|_{\alpha,\beta,1},$$

and iterating, we gain one more exponent to conclude that for all $t > 0$,

$$\|S_t f\| \lesssim \frac{1}{(t+1)^{n-}} \|f\|.$$

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