Approximating the Sparsest k-Subgraph in Chordal Graphs

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Sparsest k-Subgraph Problem (SkS)

Input: a simple undirected graph $G = (V, E)$, $k \leq |V|$.
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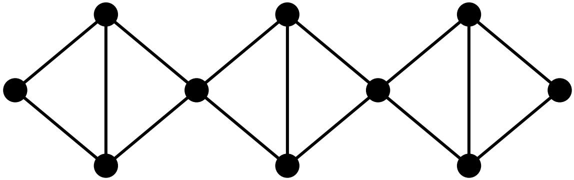

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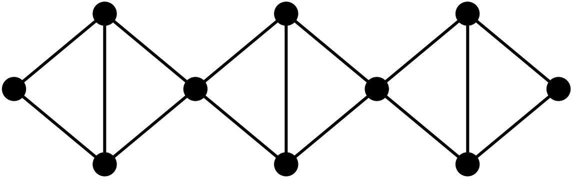

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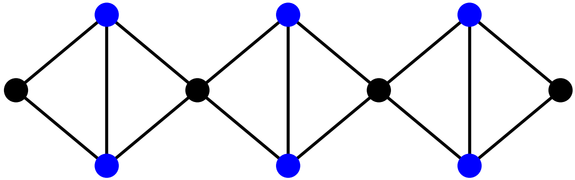

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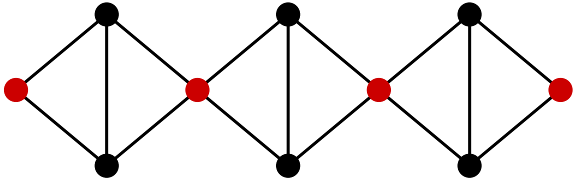

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$k = 4$: sparsest 4-subgraph = 0 edges

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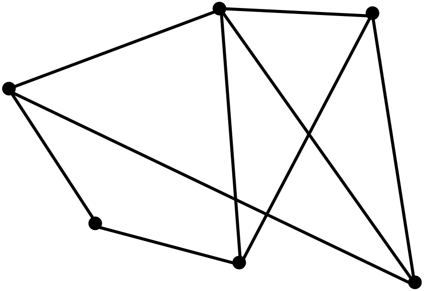
A graph G is chordal if it does not contain any cycle of length four or more as an induced subgraph.



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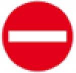

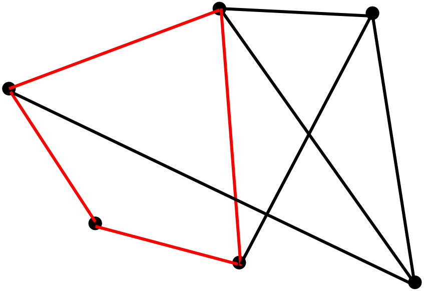

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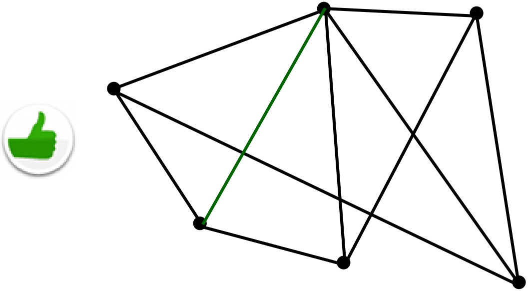

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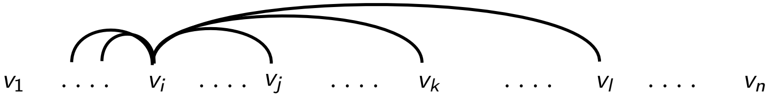

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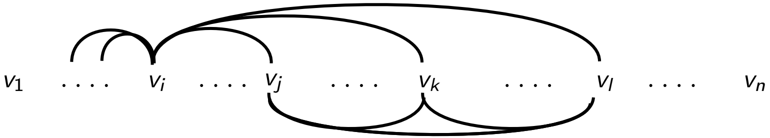

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Such an ordering can be found in polynomial time.

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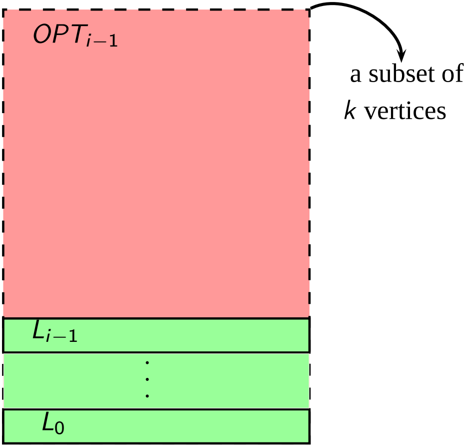
- restructuration $\mathbf{OPT}_{i-1} \rightarrow \mathbf{OPT}_i$:
  suppose that layers $L_0, \ldots, L_{i-1}$ are the same for $\mathbf{OPT}_{i-1}$ and **ALGO**

2-approximation in chordal graphs: idea of restructuration



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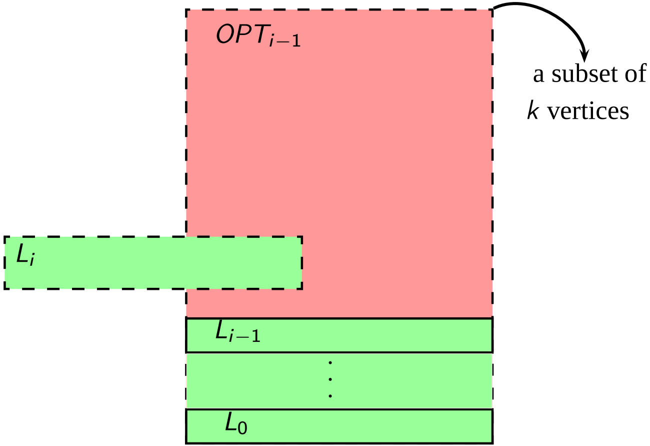

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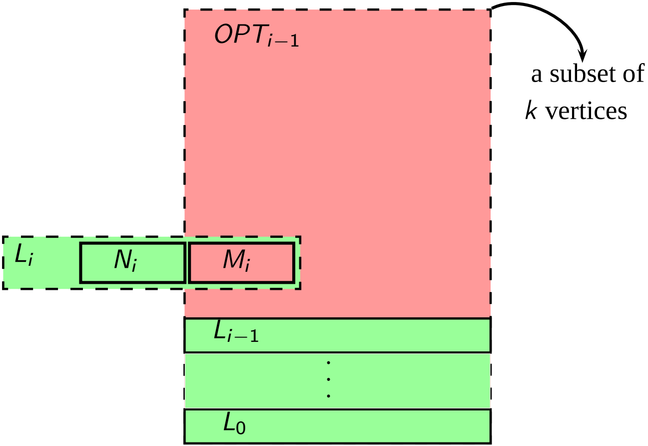

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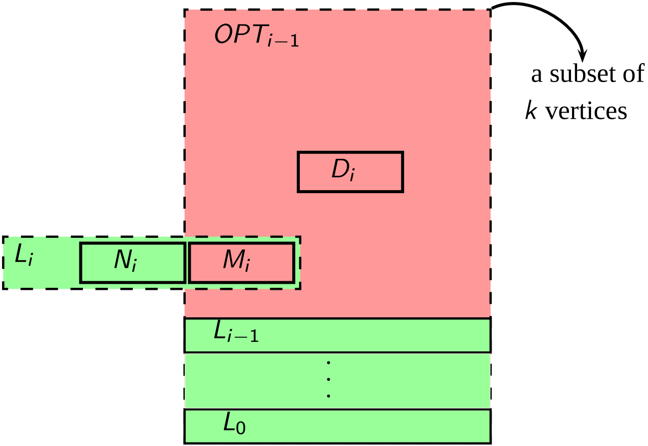

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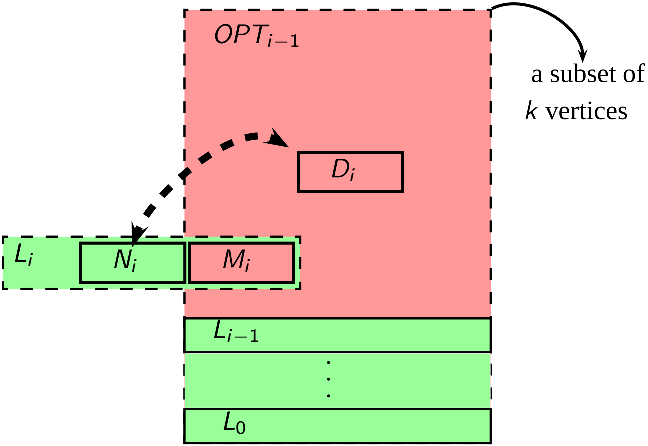

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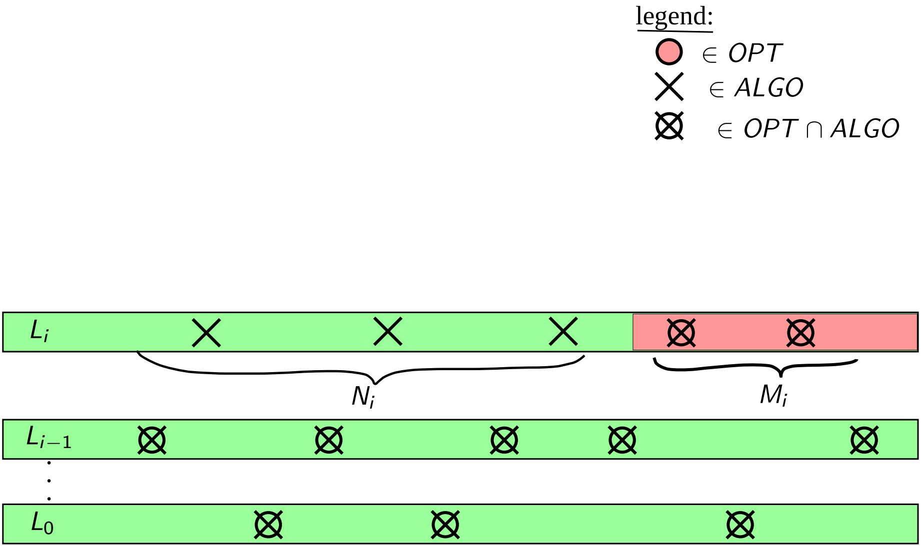

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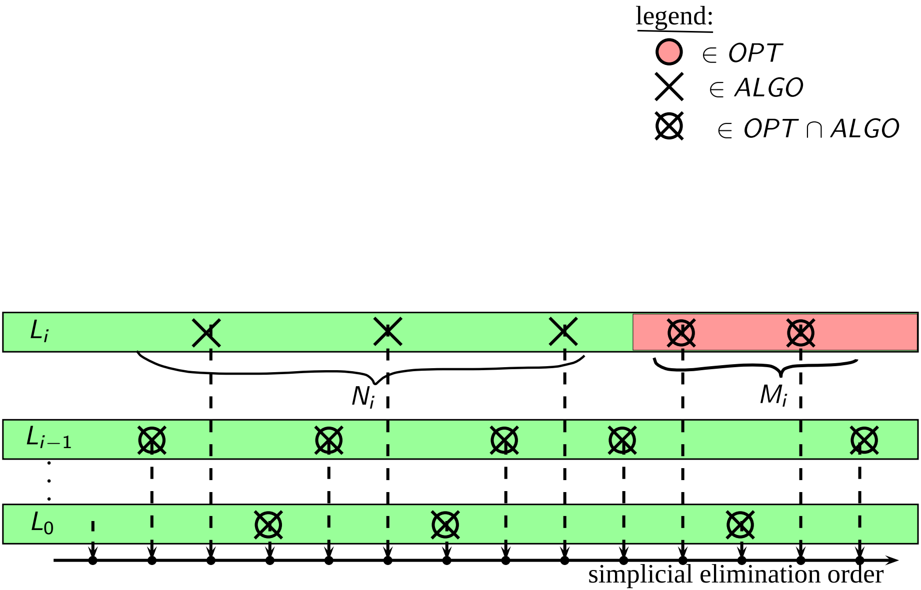

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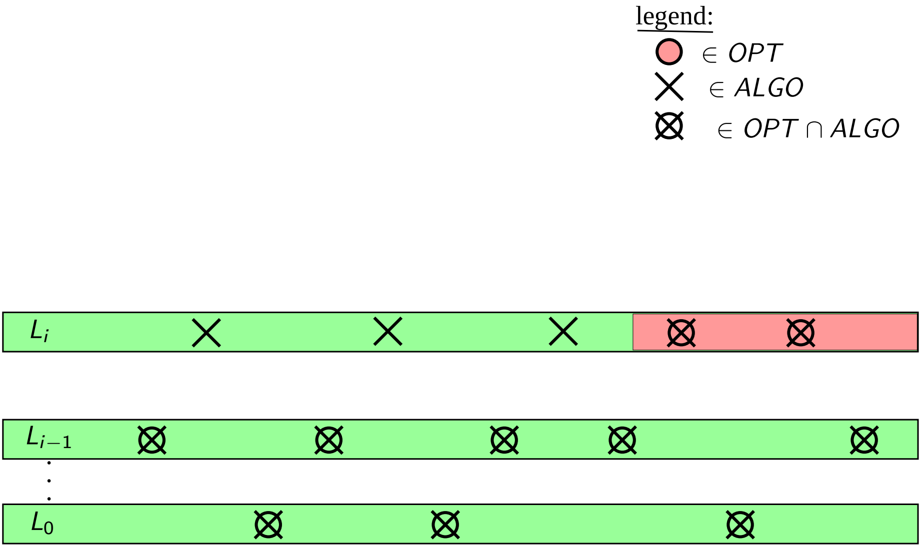

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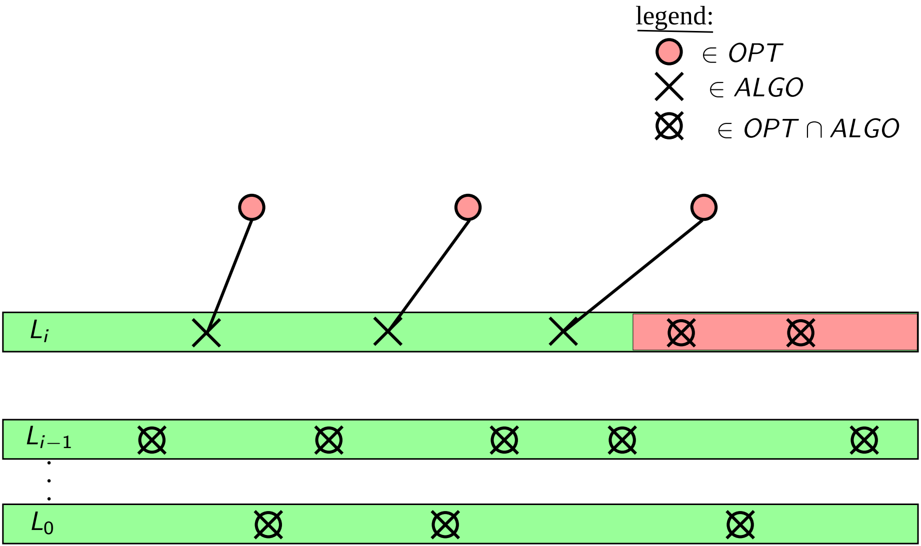

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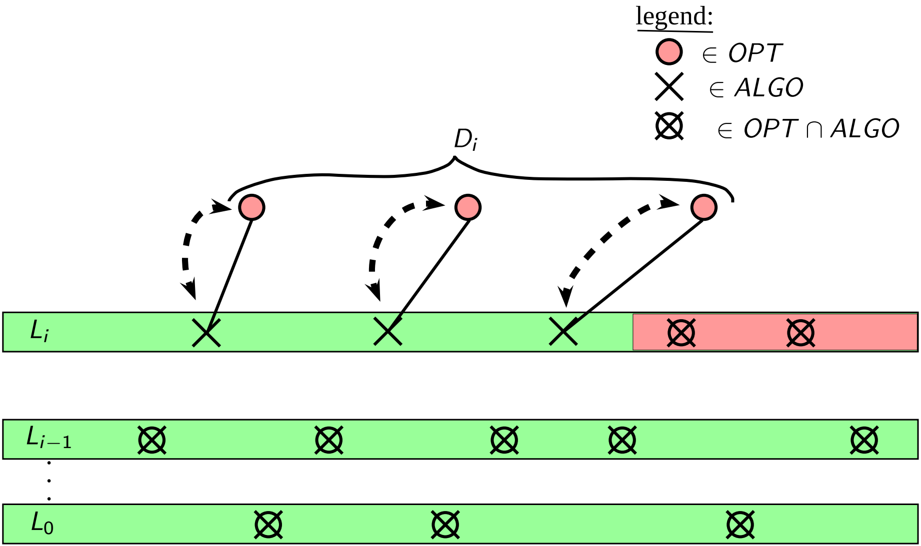

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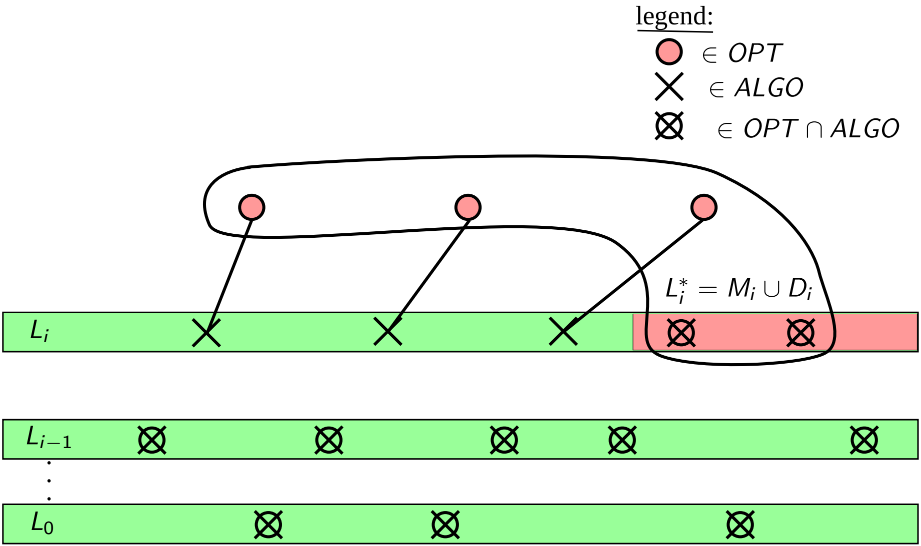

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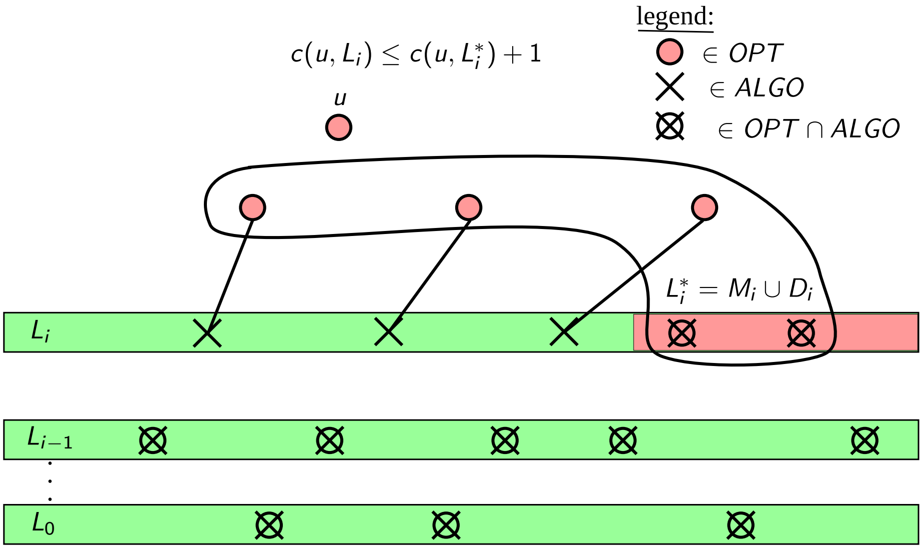

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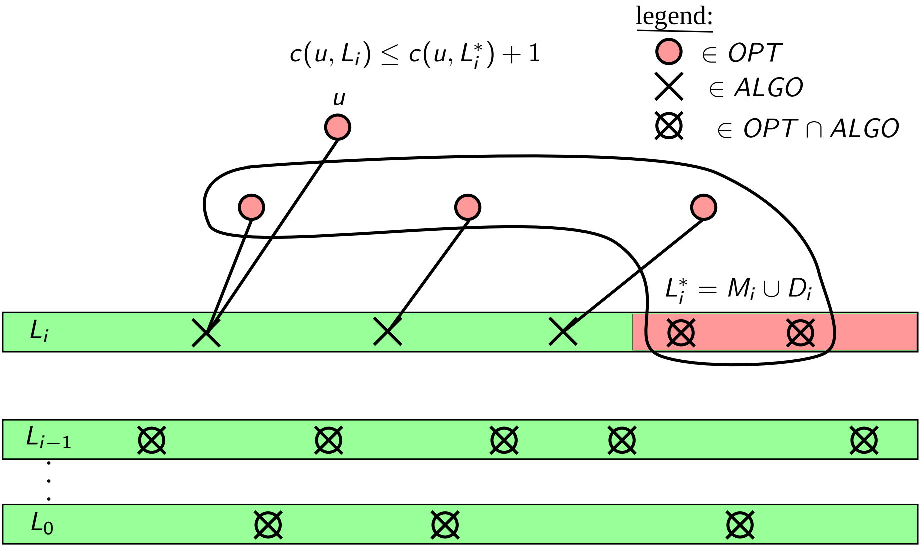

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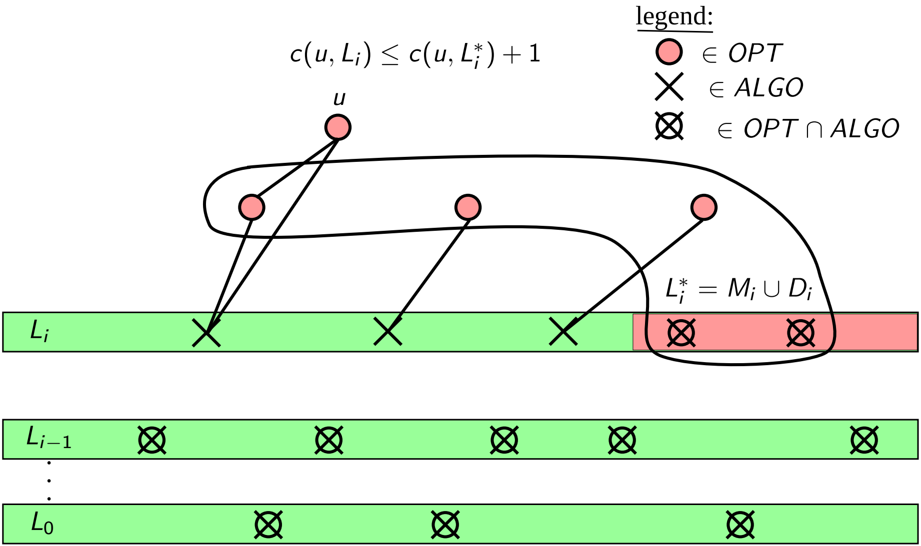

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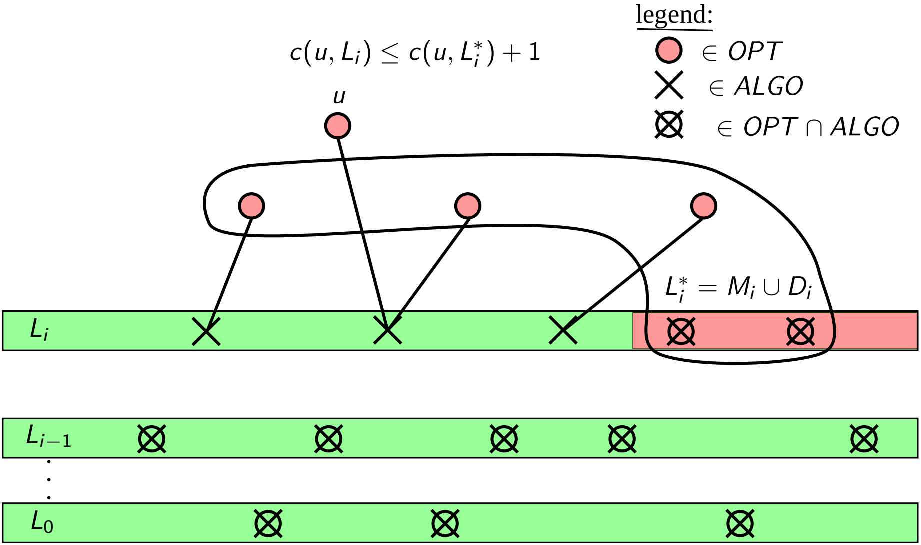
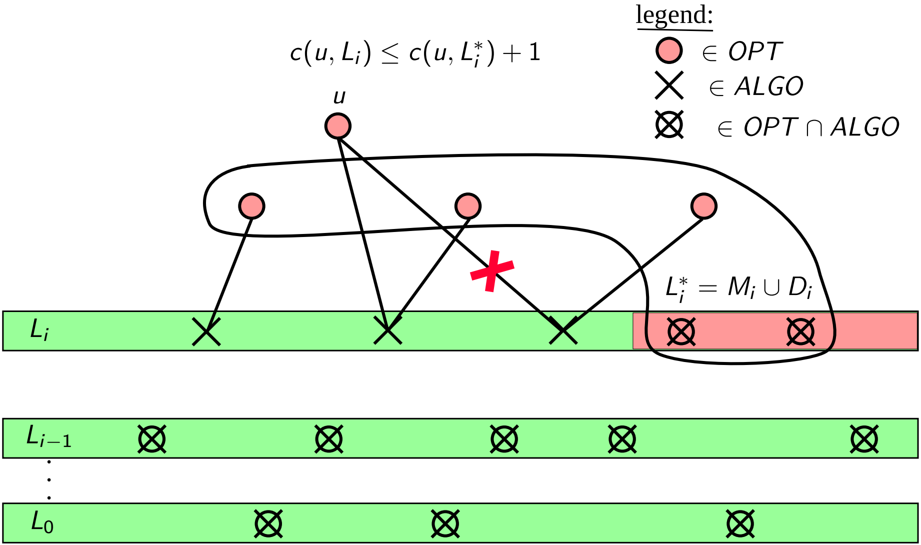
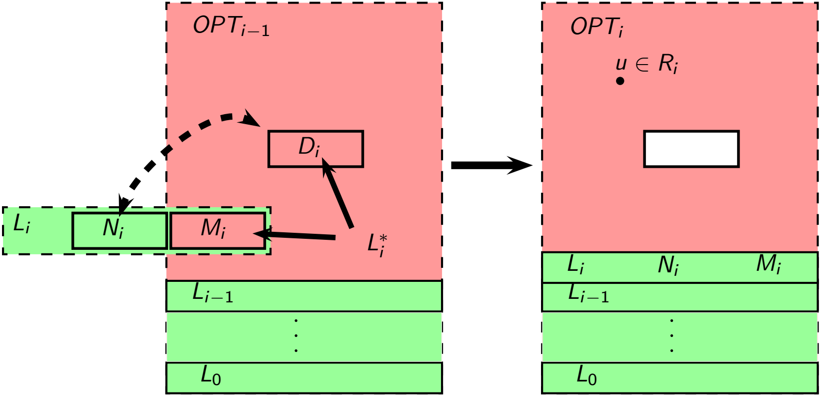
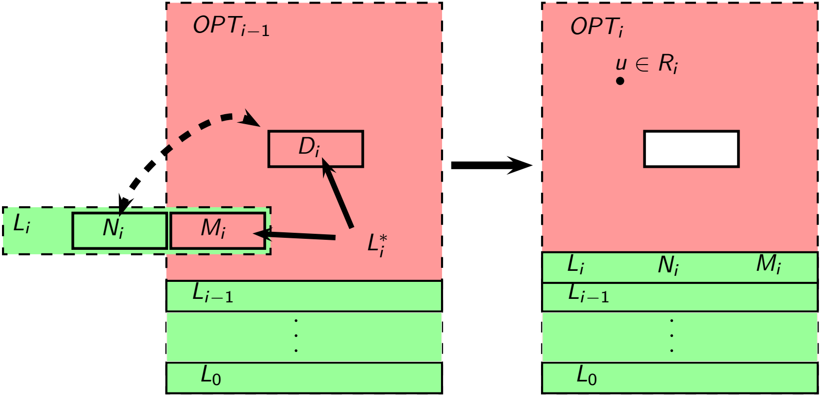


2-approximation in chordal graphs

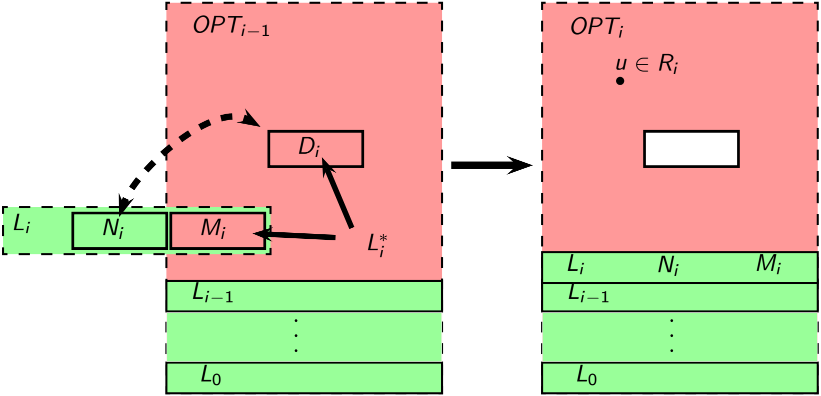




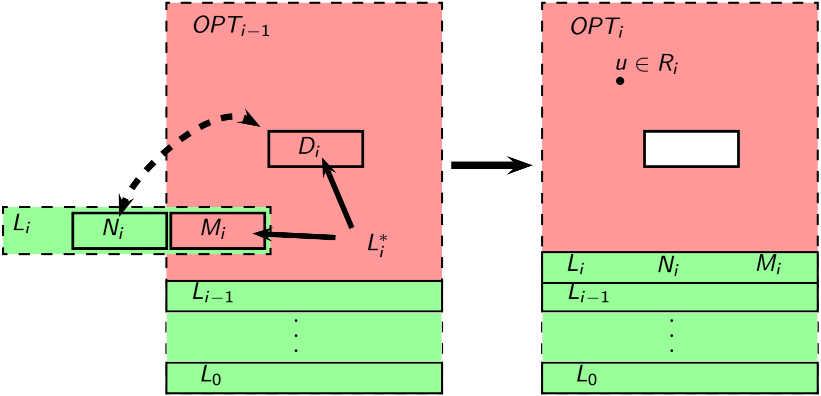
$\forall u \in R_i$ we have $c(u, L_i) \leq c(u, L_i^*) + 1$



$\forall u \in R_i$ we have $c(u, L_i) \leq c(u, L_i^*) + 1 \leq_{hyp} 2 \cdot c(u, L_i^*)$



$\forall u \in R_i$ we have $c(u, L_i) \leq c(u, L_i^*) + 1 \leq_{hyp} 2 \cdot c(u, L_i^*) \Rightarrow c(R_i, L_i) \leq 2 \cdot c(u, L_i^*)$



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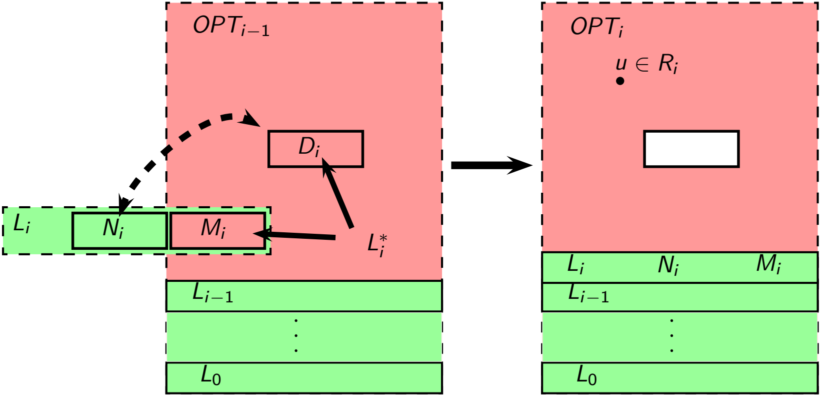
Then we can show:

$$c(OPT_i) \leq c(OPT_{i-1}) + c(R_i, L_i^*)$$



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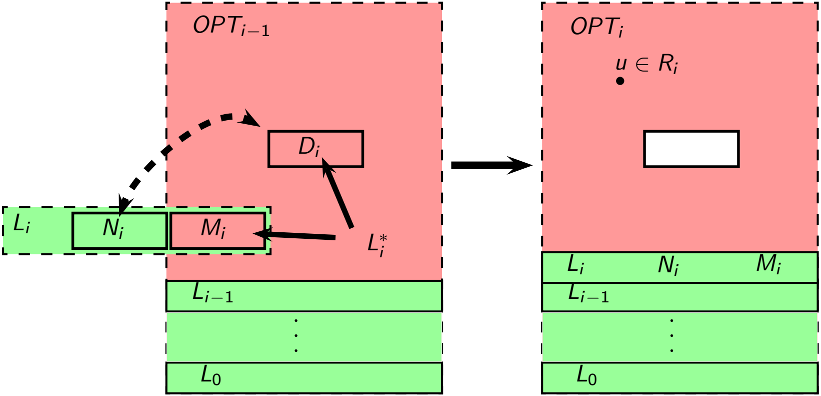
Thus:

$$c(OPT_t) \leq c(OPT_0) + \sum_{i=1}^{t} c(R_i, L_i^*)$$



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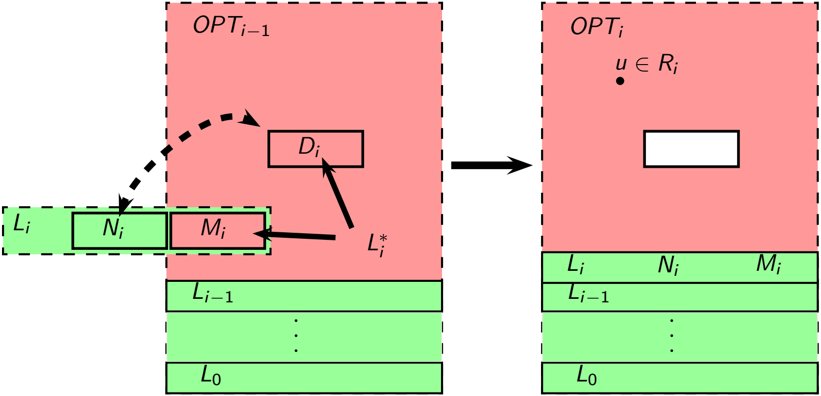
$$\Leftrightarrow c(ALGO) \leq c(OPT) + c(OPT)$$



Theorem:

There is a (tight) 2-approximation for Sparsest k-Subgraph in chordal graphs.

Conclusion

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What is known:

- SkS and DkS NP-hard in chordal graphs
- SkS and DkS approximable in chordal graphs (constant ratio)
- PTAS for DkS in interval graphs
- PTAS for SkS in proper interval graphs

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What is OPEN:

- approximation schemes/lower bounds in chordal graphs ?
- complexity status in (proper) interval graphs for SkS and DkS ?
- complexity status in planar graphs for DkS ?

Merci de votre attention !

Questions ?