Sum-Max Graph Partitioning Problem

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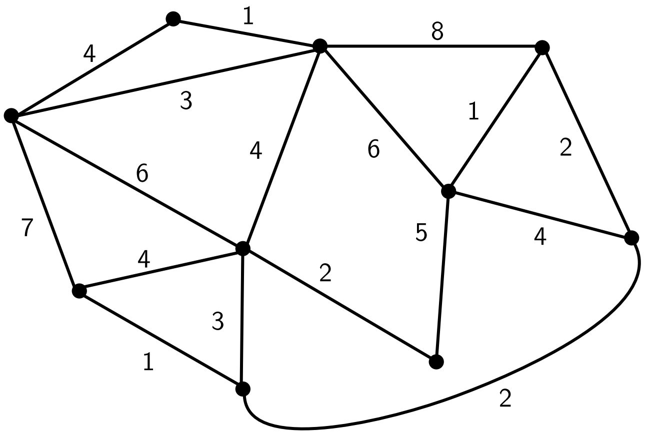

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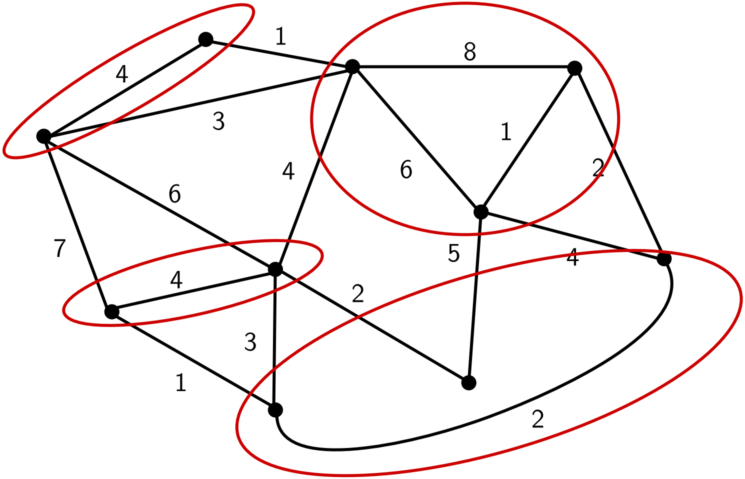

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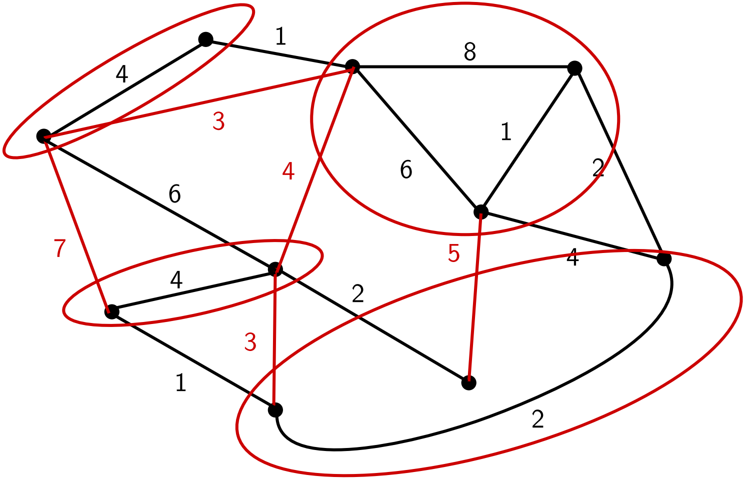

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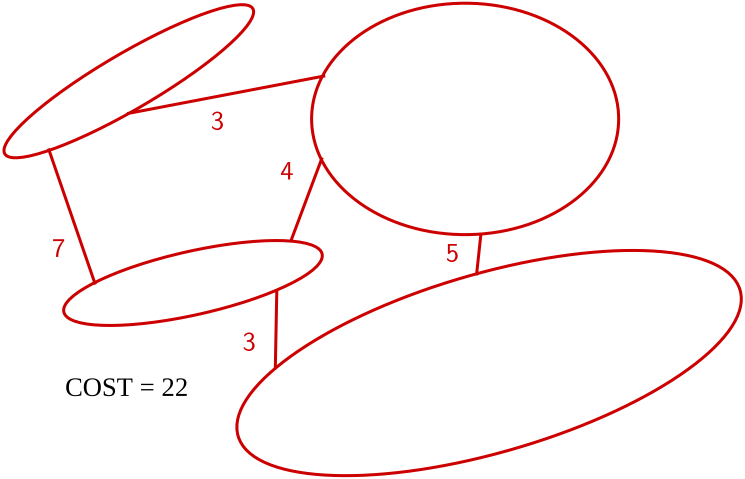

SUM-MAX GRAPH PARTITIONING

Input: a connected graph $G = (V, E)$, $w : E \to \mathbb{N}$, $k \in \mathbb{N}$
Output: a k-partition $(V_1, ..., V_k)$ of V

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- greedy $\frac{k}{2}$-approximation algorithm
- exact solutions ($k = 3$ and extensions)
- partial results and future work

Contents

Unweighted version of the problem

$w(e) = 1 \ \forall e \in E$

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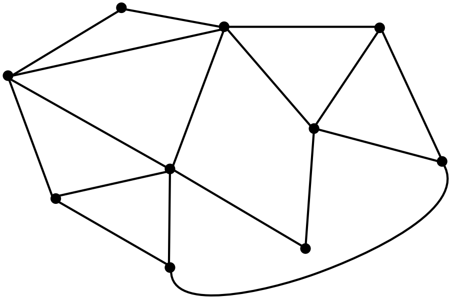

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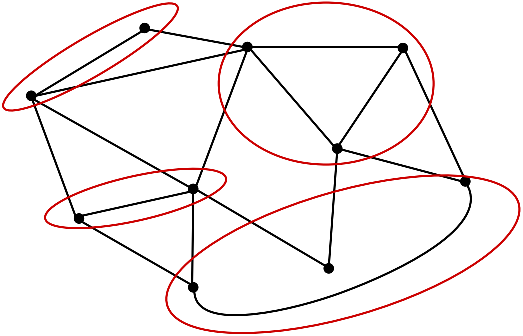

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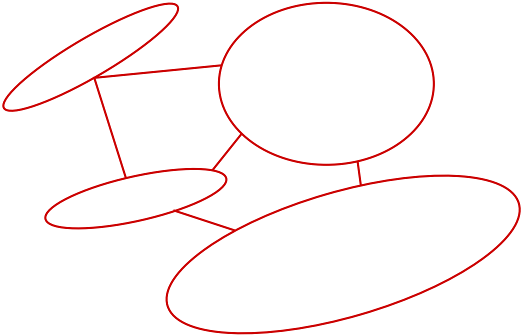

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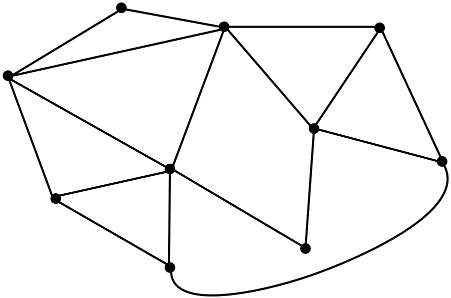

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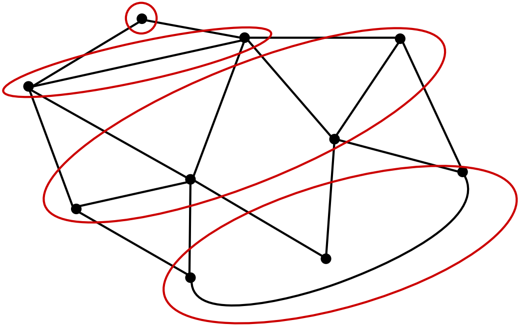

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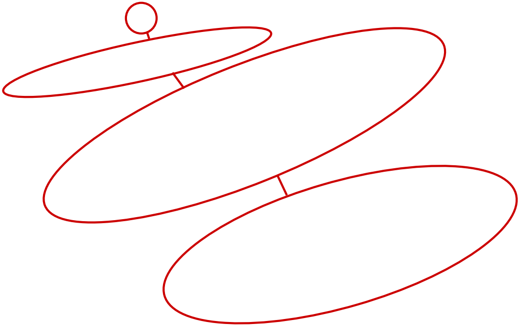

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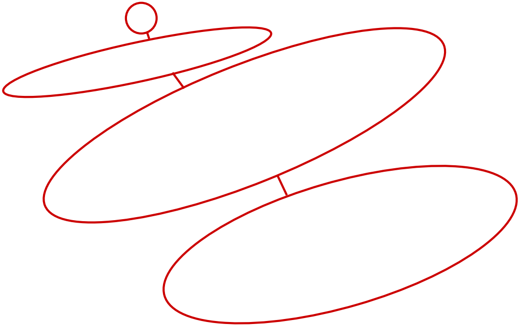

Property

For a connected unweighted graph, any optimal solution has a cost of at least $k-1$ (in this case the quotient graph is a tree)

NP, $W[1]$ hardnesses, inapproximability

Reduction from INDEPENDENT SET:

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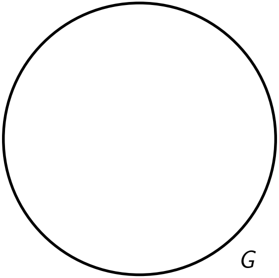
Reduction from INDEPENDENT SET:

graph G, $k \in \mathbb{N}$	$\rightarrow$	$G \cup \{\omega\}$, $(k+1)$-partition
$\alpha(G) \geq k$?		$(k+1)$-partition of cost k ?



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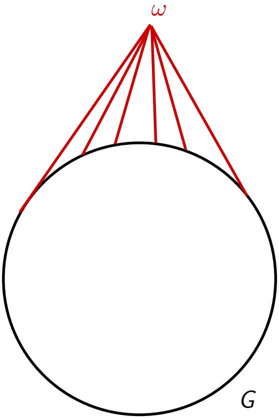

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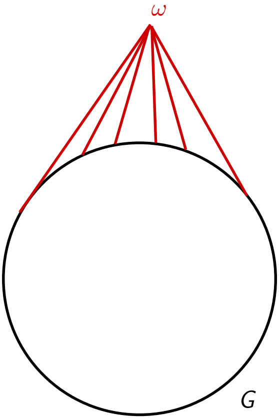

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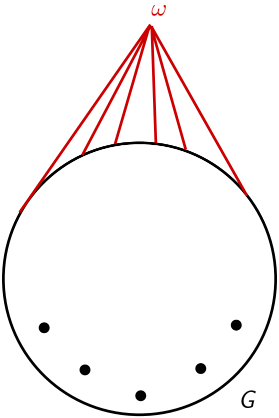

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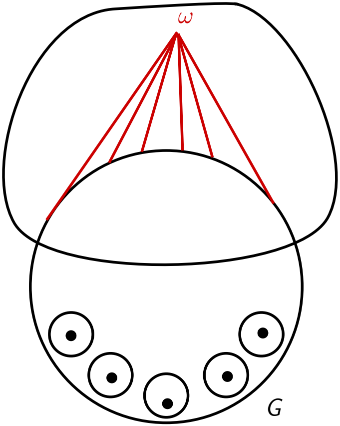

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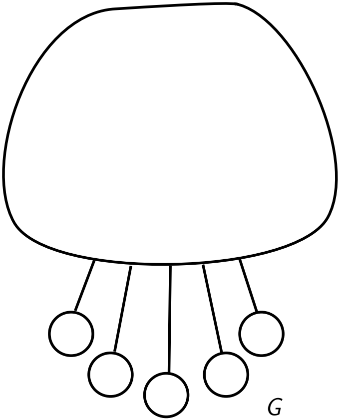

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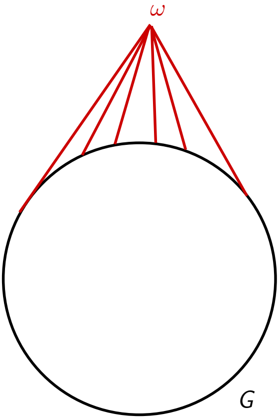

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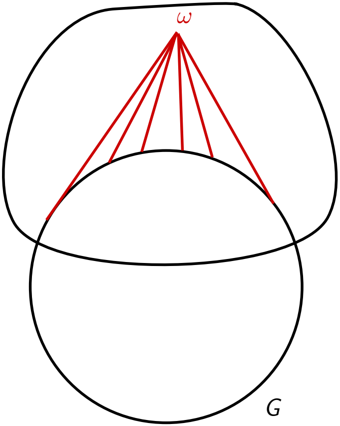

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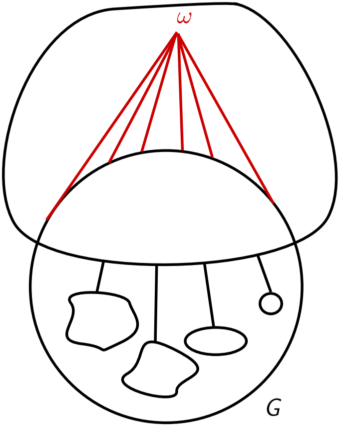

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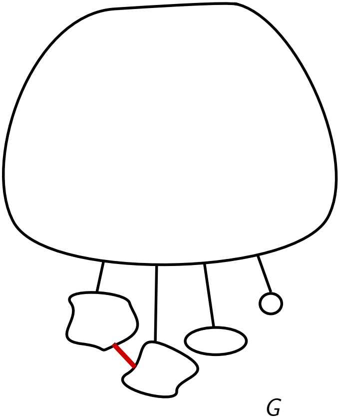

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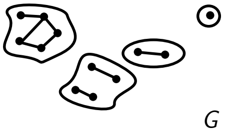

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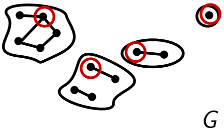
independent set of size k



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Reduction from INDEPENDENT SET:
Conclusion:

- $\alpha(G) \geq k \Rightarrow OPT(G') \leq k$
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SUM-MAX GRAPH PARTITIONING is $\mathcal{NP}$-hard, and even $W[1]$-hard for the parameter k

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We can also prove a gap preserving reduction :

- $\alpha(G) \geq k \Rightarrow OPT(G') \leq k$
- $\alpha(G) < r.k \Rightarrow OPT(G') > \frac{1}{2r}k$ for all $r << 1$

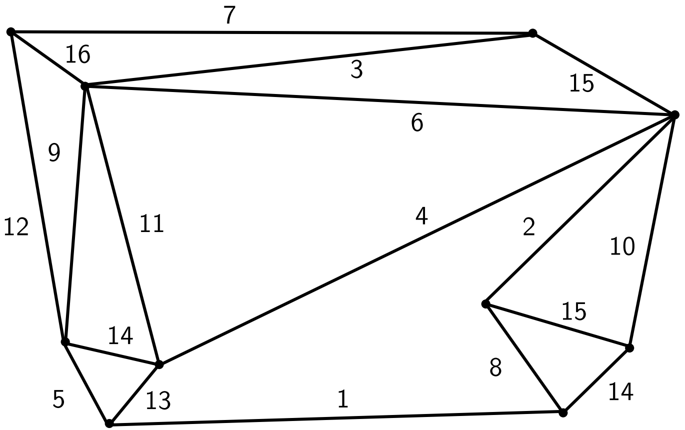
Theorem

SUM-MAX GRAPH PARTITIONING is $O(n^{1-\epsilon})$ non-approximable unless $\mathcal{P} = \mathcal{NP}$

Contents

Example, $k = 3$

Algorithm = remove the lightest edge of G until G has k connected components



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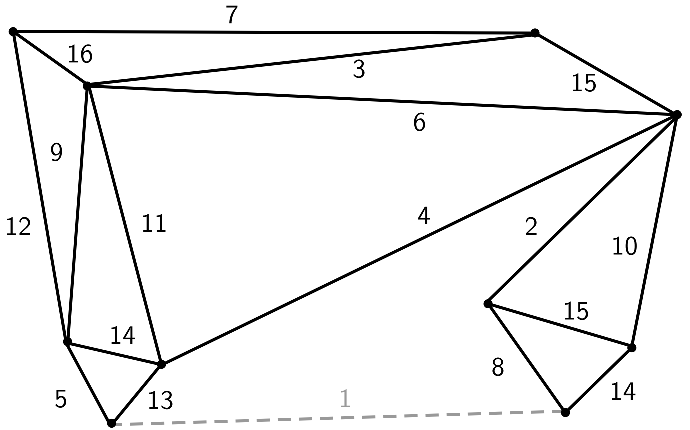

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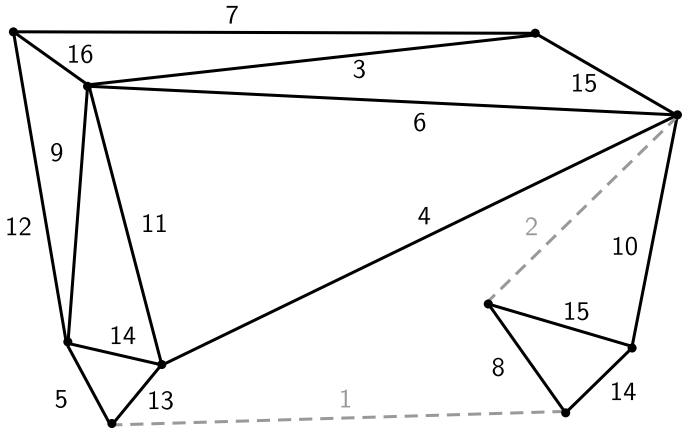

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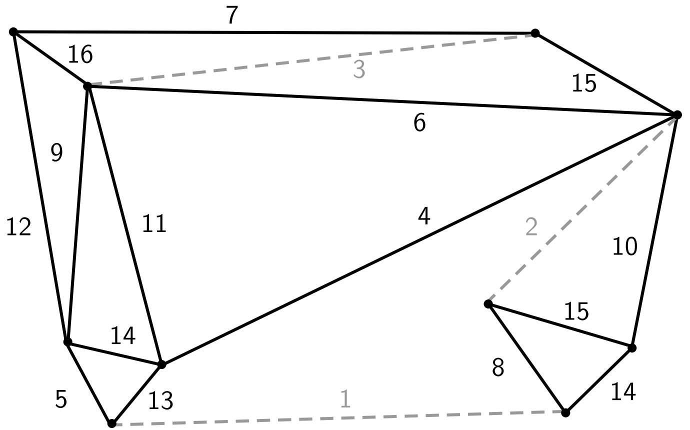

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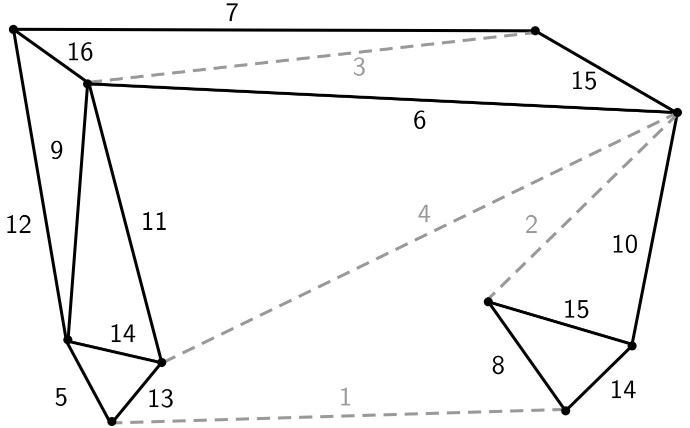

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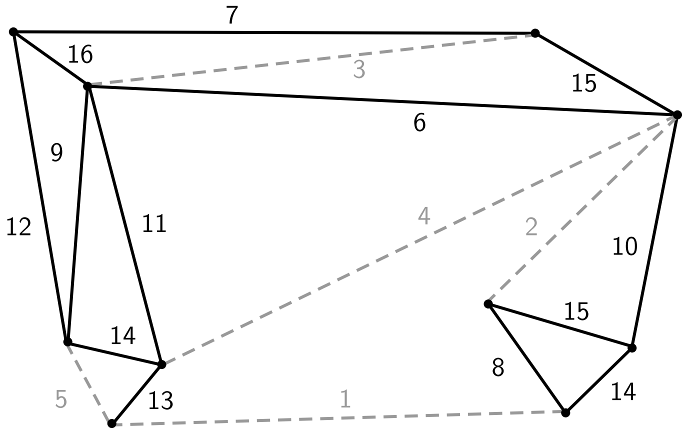

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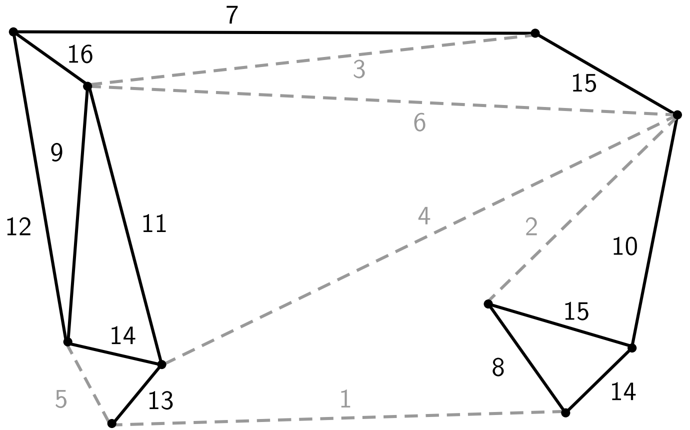

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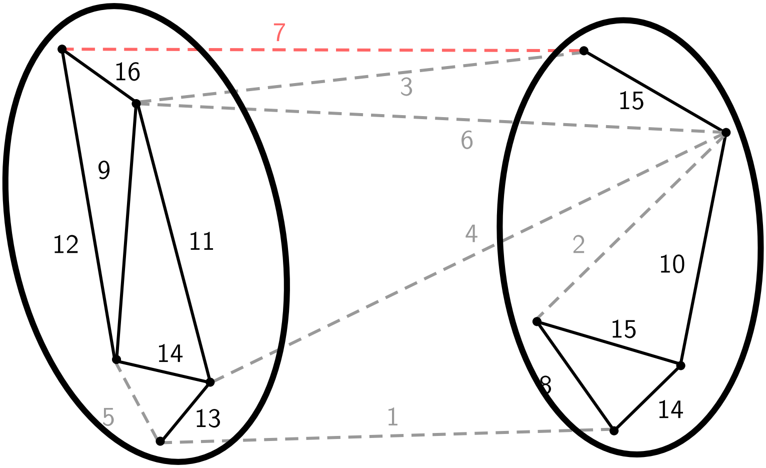

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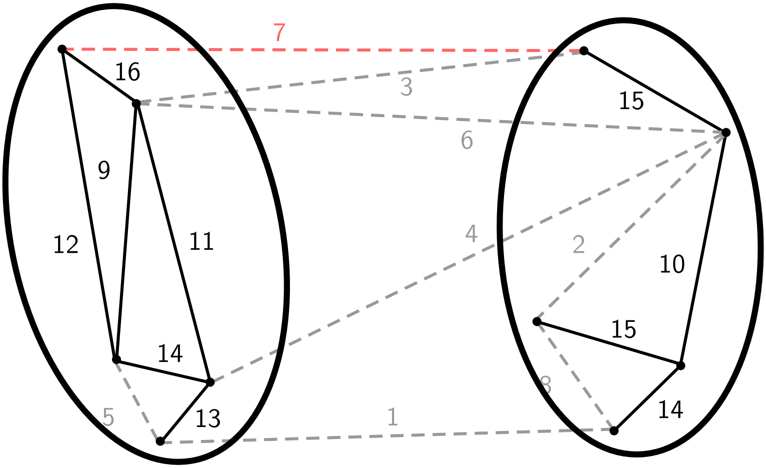

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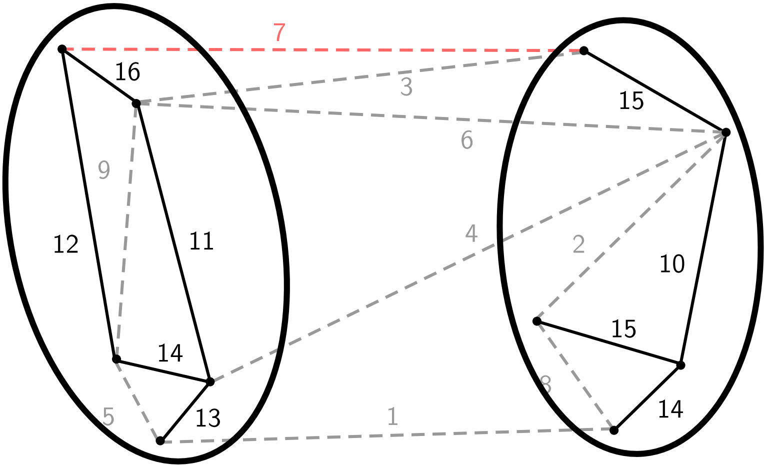

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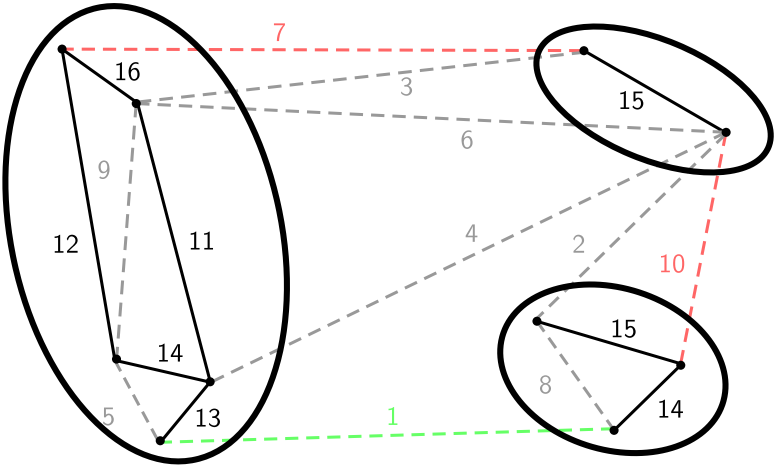

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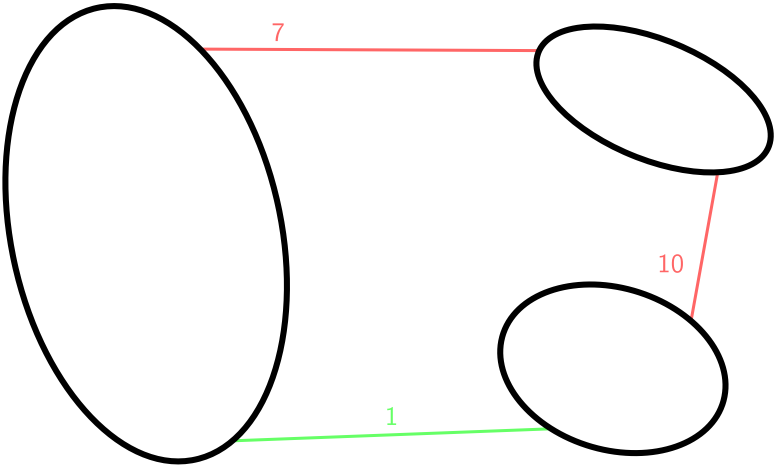

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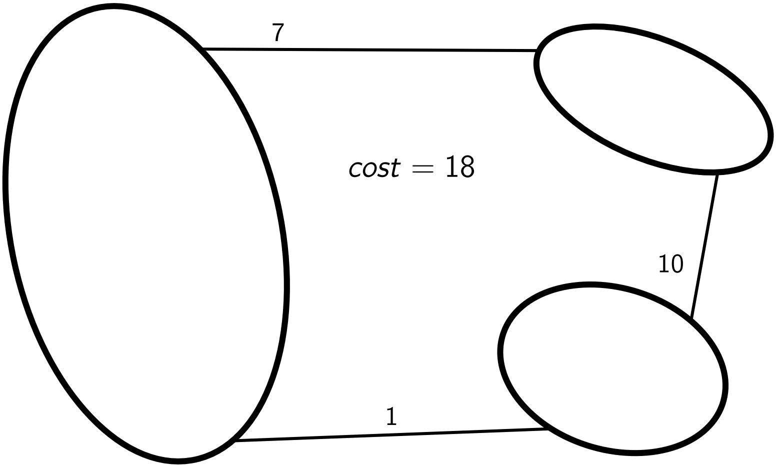

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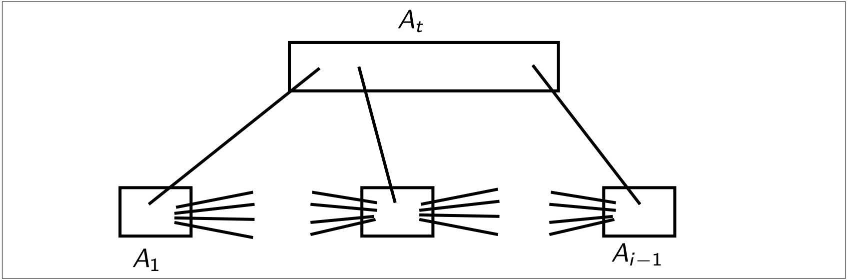

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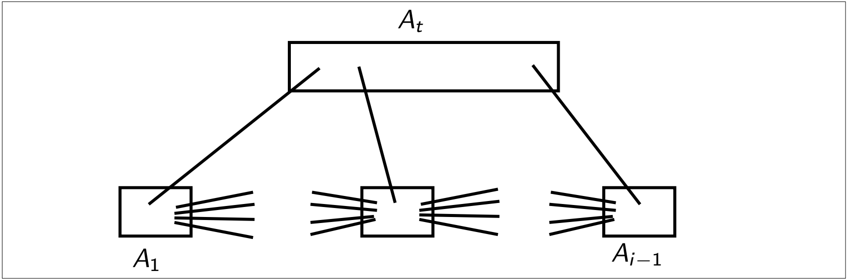
For i from 1 to $k-1$ do:

while G has i connected components do:



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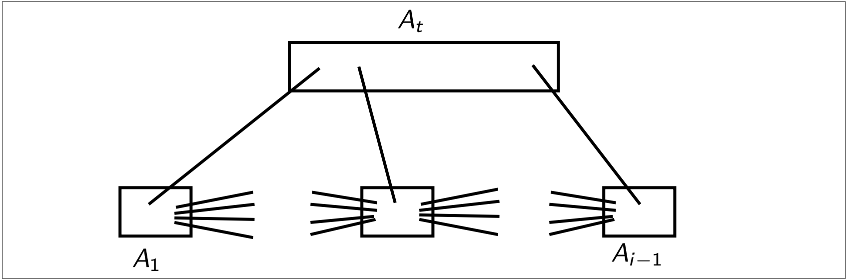

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    end while. *// let w_i be the weight of the last removed edge*

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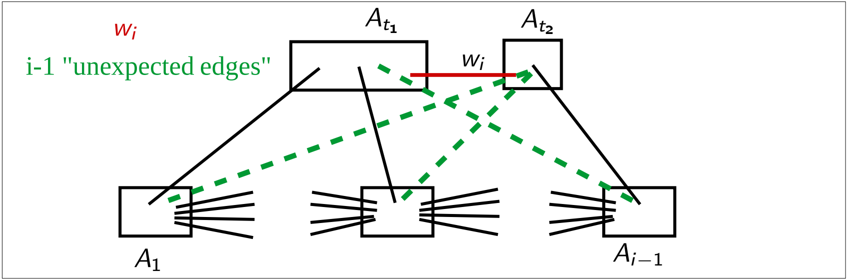

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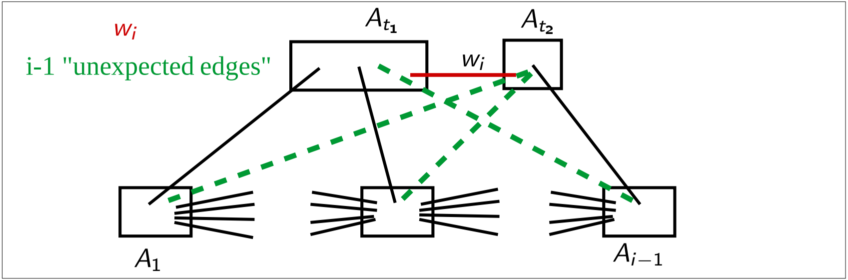

At the end:

$$\text{Solution value} = \sum_{i=1}^{k-1} w_i + \sum \text{unexpected edges}$$

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Lemma 1

At each step i: sum of edges of maximum weight outgoing from each cluster is smaller than $\sum_{j=1}^{i-1} w_j$

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For i from 1 to $k-1$ do:

    while G has i connected components do:

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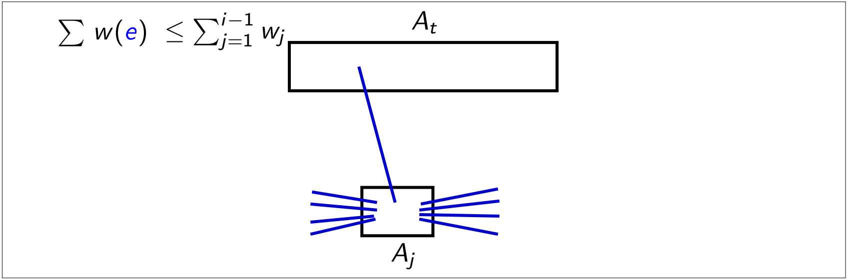

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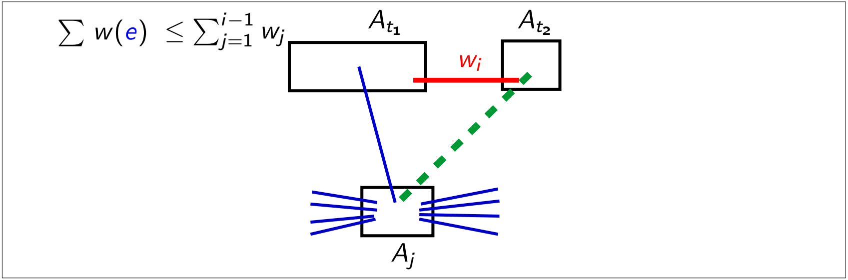

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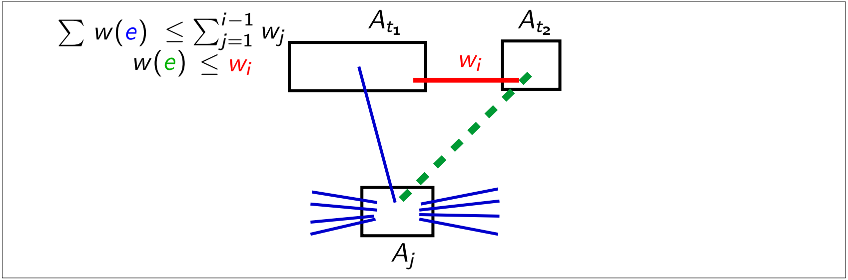

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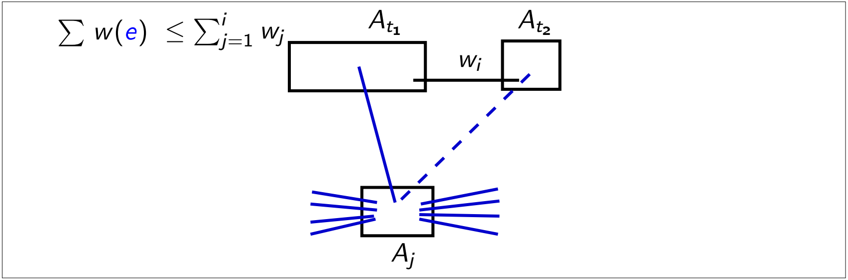

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(can be improved using the gap between edge weights)

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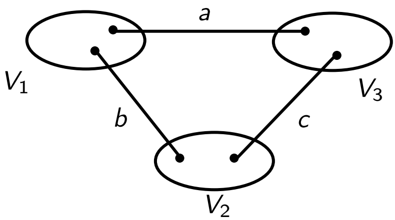

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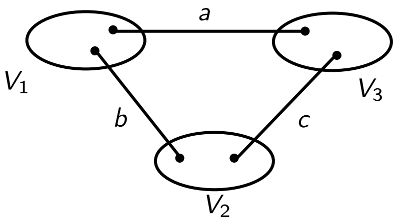

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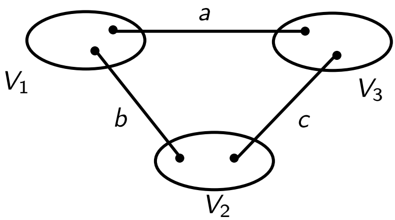

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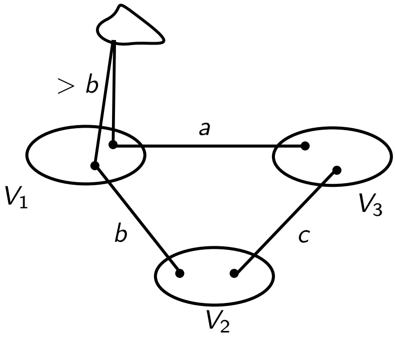

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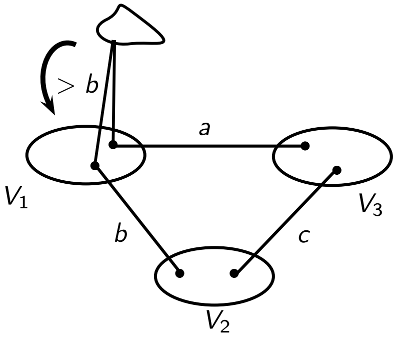

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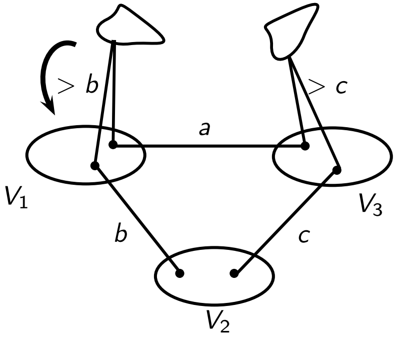

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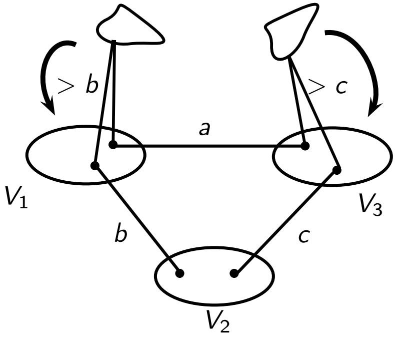

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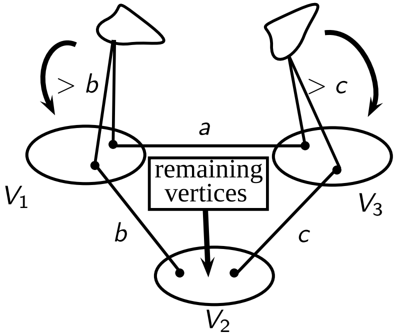

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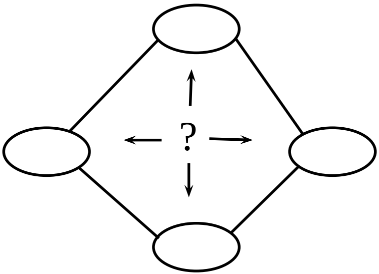
arranging vertices is $\mathcal{NP}$-complete if the quotient graph is C_4

(reduction from 3-COLORING)



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- applications in software engineering : adding/relaxing constraints
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Thank you for your attention!