On Finding a Sparse Subgraph in Subclasses of Perfect Graphs

Rémi Watrigant, Marin Bougeret and Rodolphe Giroudeau

LIRMM, Montpellier, France

1ères journées du GT CoA - Complexité et Algorithmes

Contents

k-Sparsest Subgraph Problem (k-SS)

Input: a graph $G = (V, E)$, $k \leq |V|$.
Output: a set $S \subseteq V$ of size exactly k.
Goal: minimize $|E(S)|$ (the number of edges induced by S)

k-Sparsest Subgraph Problem (k-SS)

Input: a graph $G = (V, E)$, $k \leq |V|$.
Output: a set $S \subseteq V$ of size exactly k.
Goal: minimize $|E(S)|$ (the number of edges induced by S)

- generalization of INDEPENDENT SET
  $\Rightarrow$ k-SS NP-hard in general graphs (+ no FPT, approximation algorithm)

k-Sparsest Subgraph Problem (k-SS)

Input: a graph $G = (V, E)$, $k \leq |V|$.
Output: a set $S \subseteq V$ of size exactly k.
Goal: minimize $|E(S)|$ (the number of edges induced by S)

- generalization of INDEPENDENT SET
  $\Rightarrow$ k-SS NP-hard in general graphs (+ no FPT, approximation algorithm)
  But INDEPENDENT SET is polynomial in perfect graphs!
  what about k-SS ?

k-Sparsest Subgraph Problem (k-SS)

Input: a graph $G = (V, E)$, $k \leq |V|$.
Output: a set $S \subseteq V$ of size exactly k.
Goal: minimize $|E(S)|$ (the number of edges induced by S)

- generalization of INDEPENDENT SET
  $\Rightarrow$ k-SS NP-hard in general graphs (+ no FPT, approximation algorithm)
  But INDEPENDENT SET is polynomial in perfect graphs!
  what about k-SS ?
- maximization version: k-Densest Subgraph:

k-Sparsest Subgraph Problem (k-SS)

Input: a graph $G = (V, E)$, $k \leq |V|$.
Output: a set $S \subseteq V$ of size exactly k.
Goal: minimize $|E(S)|$ (the number of edges induced by S)

- generalization of INDEPENDENT SET
  $\Rightarrow$ k-SS NP-hard in general graphs (+ no FPT, approximation algorithm)
  But INDEPENDENT SET is polynomial in perfect graphs!
  what about k-SS ?
- maximization version: k-Densest Subgraph:
  - NP-hard in chordal graphs [Corneil and Perl, 1984]
  - unknown in (proper) interval graphs (longstanding open problem) [CP84]
  - PTAS in interval graphs [Nonner, 2011]
  - constant approximation algorithm in chordal graphs [Liazi et al., 2008]

k-Sparsest Subgraph Problem (k-SS)

Input: a graph $G = (V, E)$, $k \leq |V|$.
Output: a set $S \subseteq V$ of size exactly k.
Goal: minimize $|E(S)|$ (the number of edges induced by S)

- generalization of INDEPENDENT SET
  $\Rightarrow$ k-SS NP-hard in general graphs (+ no FPT, approximation algorithm)
  But INDEPENDENT SET is polynomial in perfect graphs!
  what about k-SS ?

- maximization version: k-Densest Subgraph:
  - NP-hard in chordal graphs [Corneil and Perl, 1984]
  - unknown in (proper) interval graphs (longstanding open problem) [CP84]
  - PTAS in interval graphs [Nonner, 2011]
  - constant approximation algorithm in chordal graphs [Liazi et al., 2008]

- k-SS polynomial in:
  - split graphs (obvious)
  - bounded cliquewidth ($\Rightarrow$ trees, cographs, ...) [Boersma et al., 2012]

Recall that *proper interval* $\subset$ *interval* $\subset$ *chordal* $\subset$ *perfect*
split $\subset$ *chordal* $\subset$ *perfect*

k-Sparsest Subgraph Problem (k-SS)

Input: a graph $G = (V, E)$, $k \leq |V|$.
Output: a set $S \subseteq V$ of size exactly k.
Goal: minimize $|E(S)|$ (the number of edges induced by S)

Our results [W., Bougeret, Giroudeau, 2012]:

k-Sparsest Subgraph Problem (k-SS)

Input: a graph $G = (V, E)$, $k \leq |V|$.
Output: a set $S \subseteq V$ of size exactly k.
Goal: minimize $|E(S)|$ (the number of edges induced by S)

Our results [W., Bougeret, Giroudeau, 2012]:

- FPT algorithm in interval graphs (parameterized by the cost of the solution)
- PTAS in proper interval graphs
- *NP*-hardness in chordal graphs

k-Sparsest Subgraph Problem (k-SS)

Input: a graph $G = (V, E)$, $k \leq |V|$.
Output: a set $S \subseteq V$ of size exactly k.
Goal: minimize $|E(S)|$ (the number of edges induced by S)

Our results [W., Bougeret, Giroudeau, 2012]:

- FPT algorithm in interval graphs (parameterized by the cost of the solution)
- PTAS in proper interval graphs ← This talk
- NP-hardness in chordal graphs

k-Sparsest Subgraph Problem (k-SS)

Input: a graph $G = (V, E)$, $k \leq |V|$.
Output: a set $S \subseteq V$ of size exactly k.
Goal: minimize $|E(S)|$ (the number of edges induced by S)

Our results [W., Bougeret, Giroudeau, 2012]:

- FPT algorithm in interval graphs (parameterized by the cost of the solution)
- PTAS in proper interval graphs ← This talk
- *NP*-hardness in chordal graphs

Polynomial-Time Approximation Scheme (PTAS)

A *PTAS* for a minimization problem is an algorithm $\mathcal{A}_\epsilon$ such that for any fixed $\epsilon > 0$, $\mathcal{A}_\epsilon$ runs in polynomial time and outputs a solution of cost $< (1+\epsilon)OPT$

k-Sparsest Subgraph Problem (k-SS)

Input: a graph $G = (V, E)$, $k \leq |V|$.
Output: a set $S \subseteq V$ of size exactly k.
Goal: minimize $|E(S)|$ (the number of edges induced by S)

Our results [W., Bougeret, Giroudeau, 2012]:

- FPT algorithm in interval graphs (parameterized by the cost of the solution)
- PTAS in proper interval graphs ← This talk
- NP-hardness in chordal graphs

Interval graph = intersection graph of intervals in the real line.

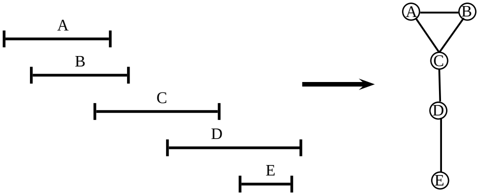
k-Sparsest Subgraph Problem (k-SS)

Input: a graph $G = (V, E)$, $k \leq |V|$.
Output: a set $S \subseteq V$ of size exactly k.
Goal: minimize $|E(S)|$ (the number of edges induced by S)

Our results [W., Bougeret, Giroudeau, 2012]:

- FPT algorithm in interval graphs (parameterized by the cost of the solution)
- PTAS in proper interval graphs ← This talk
- NP-hardness in chordal graphs

Interval graph = intersection graph of intervals in the real line.



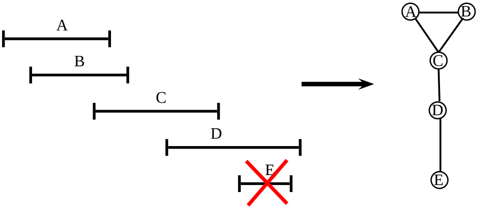
k-Sparsest Subgraph Problem (k-SS)

Input: a graph $G = (V, E)$, $k \leq |V|$.
Output: a set $S \subseteq V$ of size exactly k.
Goal: minimize $|E(S)|$ (the number of edges induced by S)

Our results [W., Bougeret, Giroudeau, 2012]:

- FPT algorithm in interval graphs (parameterized by the cost of the solution)
- PTAS in proper interval graphs ← This talk
- NP-hardness in chordal graphs

Interval graph = intersection graph of intervals in the real line.



proper interval graph = no interval contains properly another one = unit interval graphs

Contents

PTAS in Proper Interval Graphs

Idea of the algorithm:

PTAS in Proper Interval Graphs

Idea of the algorithm:

- sort intervals according to their right (or left) endpoints

PTAS in Proper Interval Graphs

Idea of the algorithm:

- sort intervals according to their right (or left) endpoints
- greedy decomposition of the graph into a path of separators/cliques

PTAS in Proper Interval Graphs

Idea of the algorithm:

- sort intervals according to their right (or left) endpoints
- greedy decomposition of the graph into a path of separators/cliques
- re-structuration of an optimal solution into a near optimal solution such that all near optimal solutions can be enumerated in polynomial time

PTAS in Proper Interval Graphs

Idea of the algorithm:

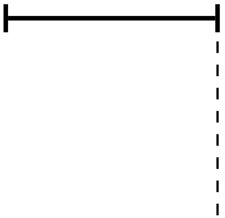
- sort intervals according to their right (or left) endpoints
- greedy decomposition of the graph into a path of separators/cliques
- re-structuration of an optimal solution into a near optimal solution such that all near optimal solutions can be enumerated in polynomial time
- dynamic programming processes the graph through the decomposition, enumerating all possible solutions.

PTAS in Proper Interval Graphs

The decomposition:

PTAS in Proper Interval Graphs

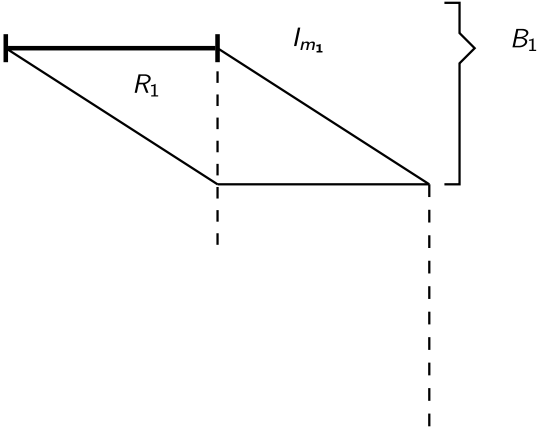
The decomposition:



I_{m_1}

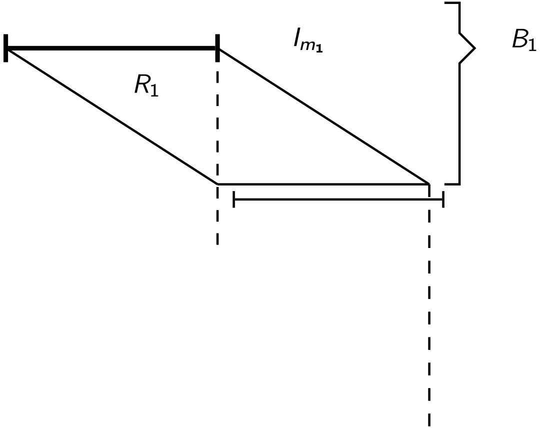
PTAS in Proper Interval Graphs

The decomposition:



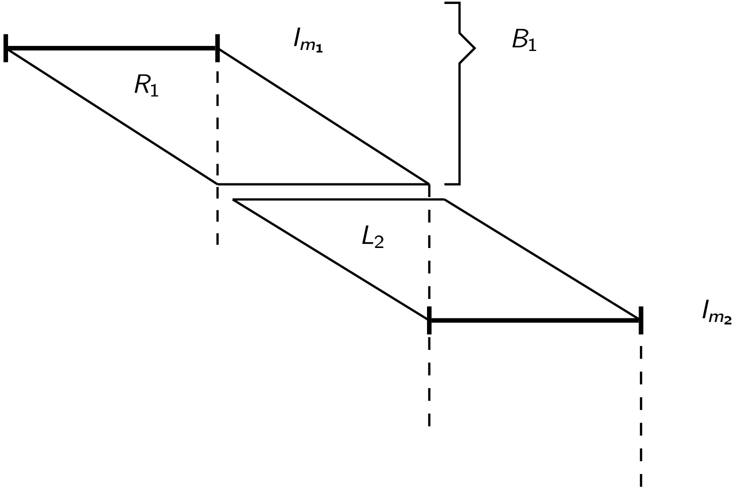
PTAS in Proper Interval Graphs

The decomposition:



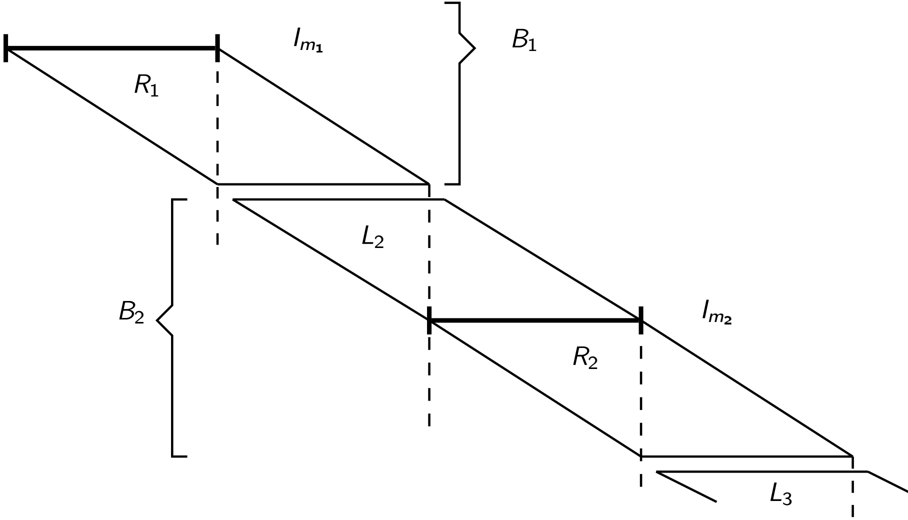
PTAS in Proper Interval Graphs

The decomposition:



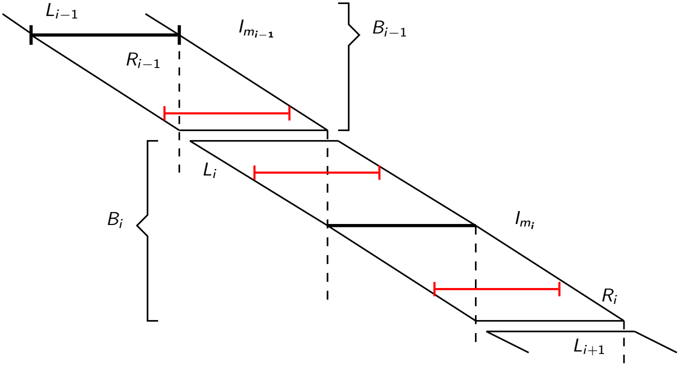
PTAS in Proper Interval Graphs

The decomposition:



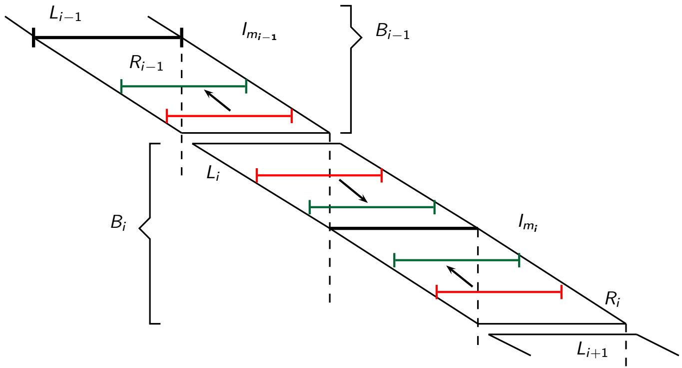
PTAS in Proper Interval Graphs

Restructuration of a solution : **compaction** $S \longmapsto comp(S)$



PTAS in Proper Interval Graphs

Restructuration of a solution : **compaction** $S \longmapsto comp(S)$



Remark

If for each block, the compaction produces a ρ-approximated solution, then it is a ρ-approximated solution for the **whole** graph.

PTAS in Proper Interval Graphs

Re-structuration of optimal solutions

Let us built a compaction that yields a $(1+\frac{4}{P})$-approximation for any fixed P.

PTAS in Proper Interval Graphs

Re-structuration of optimal solutions

Let us built a compaction that yields a $(1 + \frac{4}{P})$-approximation for any fixed P.
Let $X \subseteq B_i$ be a solution. We note $X = X_L \cup X_R$. Set sizes are in lowercase.

PTAS in Proper Interval Graphs

Re-structuration of optimal solutions

Let us built a compaction that yields a $(1 + \frac{4}{P})$-approximation for any fixed P.

Let $X \subseteq B_i$ be a solution. We note $X = X_L \cup X_R$. Set sizes are in lowercase.

- we divide X_L into P consecutive subsets of same size $q_L \rightarrow X_1^L, ..., X_P^L$
- we divide X_R into P consecutive subsets of same size $q_R \rightarrow X_1^R, ..., X_P^R$

Then define the compaction: for any $t \in \{1, ..., P\}$

PTAS in Proper Interval Graphs

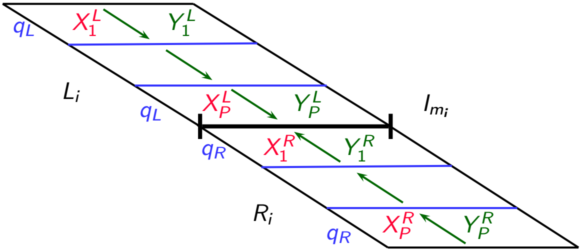
Re-structuration of optimal solutions

Let us built a compaction that yields a $(1 + \frac{4}{P})$-approximation for any fixed P.
Let $X \subseteq B_i$ be a solution. We note $X = X_L \cup X_R$. Set sizes are in lowercase.

- we divide X_L into P consecutive subsets of same size $q_L \rightarrow X_1^L, ..., X_P^L$
- we divide X_R into P consecutive subsets of same size $q_R \rightarrow X_1^R, ..., X_P^R$

Then define the compaction: for any $t \in \{1, ..., P\}$

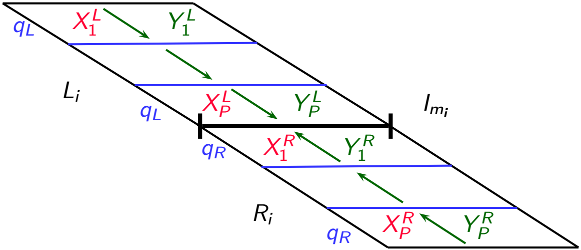
- Y_t^L are the q_L rightmost intervals of the t^{th} left block.
- Y_t^R are the q_R leftmost intervals of the t^{th} right block.



PTAS in Proper Interval Graphs

Re-structuration of optimal solutions

What do we need to construct such a solution ?



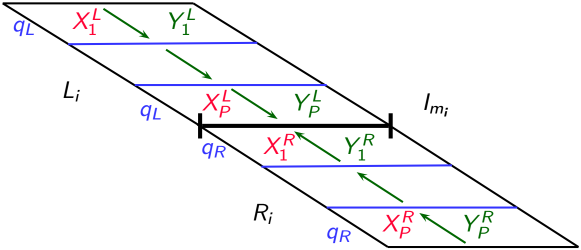
PTAS in Proper Interval Graphs

Re-structuration of optimal solutions

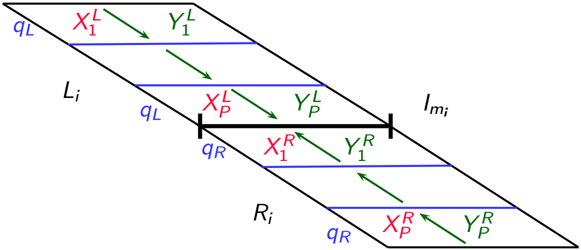
What do we need to construct such a solution ?

- the leftmost interval of the t^{th} left block for $t \in \{1, ..., P\}$
- the rightmost interval of the t^{th} right block for $t \in \{1, ..., P\}$
- x_R, x_L (plus remainders of divisions by P...)

$\Rightarrow 2P + O(1)$ variables ranging in $\{0, ..., n\}$

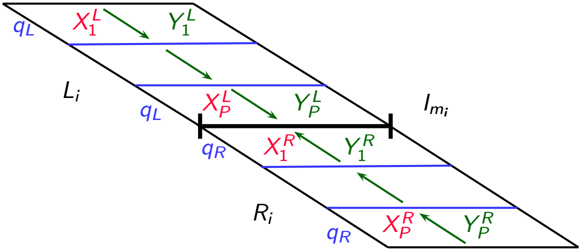


PTAS in Proper Interval Graphs



Sketch of proof of the $(1 + \frac{4}{P})$ approximation ratio:

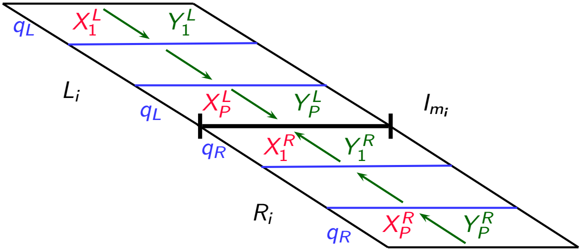
PTAS in Proper Interval Graphs



Sketch of proof of the $(1 + \frac{4}{P})$ approximation ratio:

- $SOL = \binom{x_L}{2} + \binom{x_R}{2} + \sum_{t=1}^{a} \sum_{u=1}^{a} E(Y_t^L, Y_u^R)$

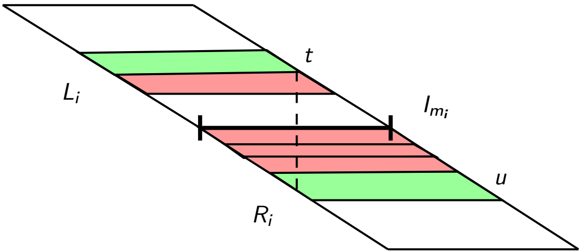
PTAS in Proper Interval Graphs



Sketch of proof of the $(1 + \frac{4}{P})$ approximation ratio:

- $SOL = \binom{x_L}{2} + \binom{x_R}{2} + \sum_{t=1}^{a} \sum_{u=1}^{a} E(Y_t^L, Y_u^R)$
- $OPT = \binom{x_L}{2} + \binom{x_R}{2} + \sum_{t=1}^{a} \sum_{u=1}^{a} E(X_t^L, X_u^R)$

PTAS in Proper Interval Graphs

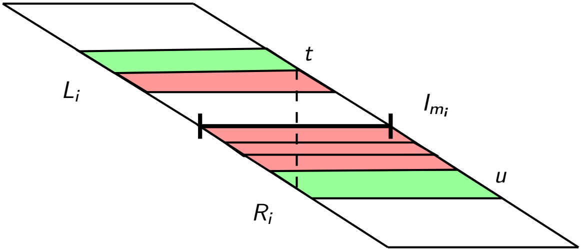


Sketch of proof of the $(1+\frac{4}{P})$ approximation ratio:

- $SOL = \binom{x_L}{2} + \binom{x_R}{2} + \sum_{t=1}^{a}\sum_{u=1}^{a} E(Y_t^L, Y_u^R)$
- $OPT = \binom{x_L}{2} + \binom{x_R}{2} + \sum_{t=1}^{a}\sum_{u=1}^{a} E(X_t^L, X_u^R)$

But:

PTAS in Proper Interval Graphs



Sketch of proof of the $(1 + \frac{4}{P})$ approximation ratio:

- $SOL = \binom{x_L}{2} + \binom{x_R}{2} + \sum_{t=1}^{a} \sum_{u=1}^{a} E(Y_t^L, Y_u^R)$
- $OPT = \binom{x_L}{2} + \binom{x_R}{2} + \sum_{t=1}^{a} \sum_{u=1}^{a} E(X_t^L, X_u^R)$

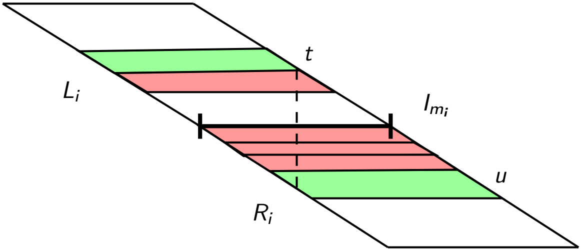
But:

- if some intervals of Y_t^L overlap some intervals of Y_u^R

Then:

- all intervals of X_{t+1}^L overlap all intervals of $\bigcup_{i=1}^{u-1} X_i^R$

PTAS in Proper Interval Graphs



Sketch of proof of the $(1 + \frac{4}{P})$ approximation ratio:

- $SOL = \binom{x_L}{2} + \binom{x_R}{2} + \sum_{t=1}^{a} \sum_{u=1}^{a} E(Y_t^L, Y_u^R)$
- $OPT = \binom{x_L}{2} + \binom{x_R}{2} + \sum_{t=1}^{a} \sum_{u=1}^{a} E(X_t^L, X_u^R)$

But:

- if some intervals of Y_t^L overlap some intervals of Y_u^R

Then:

- all intervals of X_{t+1}^L overlap all intervals of $\bigcup_{i=1}^{u-1} X_i^R$

Finally, we can prove that $\frac{SOL}{OPT} \leq 1 + \frac{4}{P}$

PTAS in Proper Interval Graphs

Conclusion:

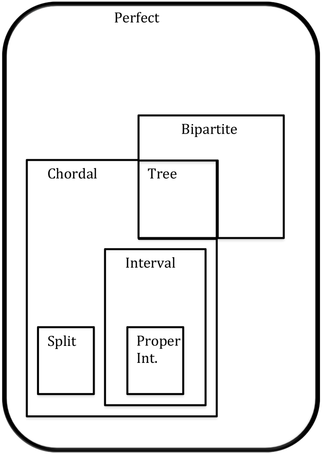
Theorem

For any P, the previous algorithm outputs a $(1 + \frac{4}{P})$-approximation for the k-Sparsest Subgraph in Proper Interval graphs in $O(n^{O(P)})$

Contents

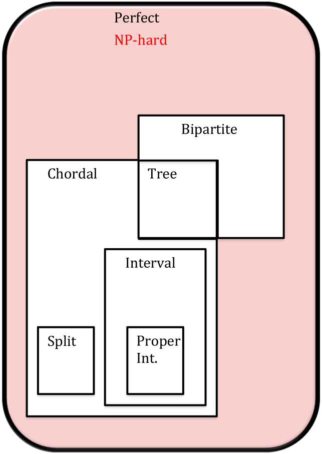
Open problems and Future Work

Complexity of k-Sparsest Subgraph:



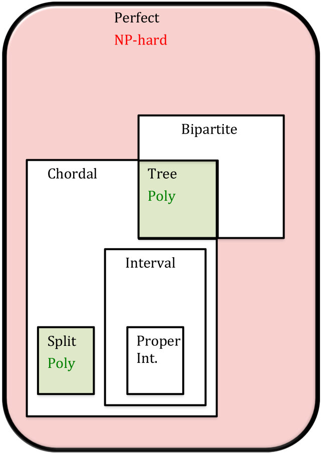
Open problems and Future Work

Complexity of k-Sparsest Subgraph:



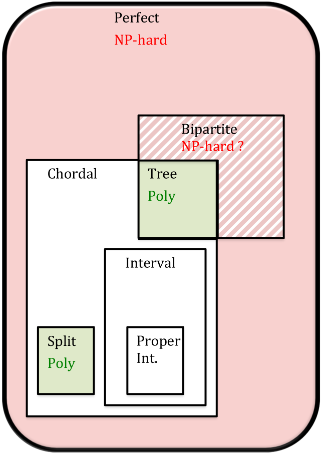
Open problems and Future Work

Complexity of k-Sparsest Subgraph:



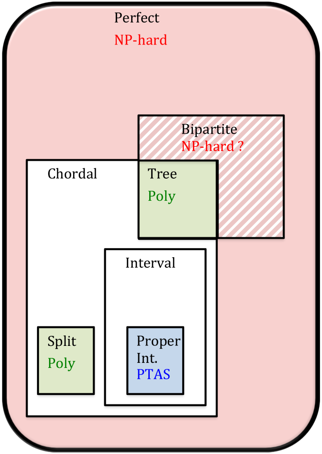
Open problems and Future Work

Complexity of k-Sparsest Subgraph:



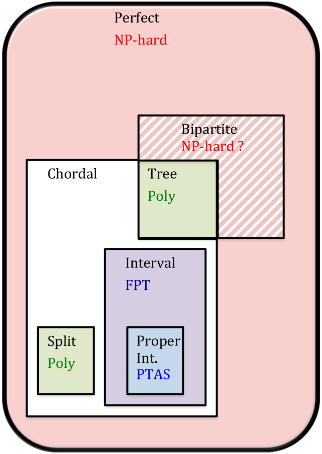
Open problems and Future Work

Complexity of k-Sparsest Subgraph:



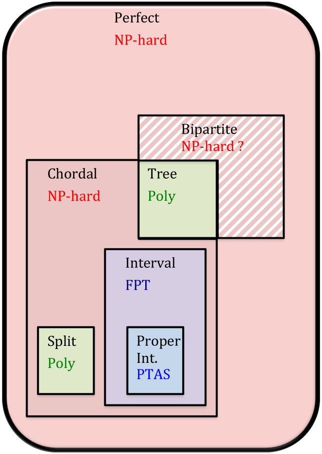
Open problems and Future Work

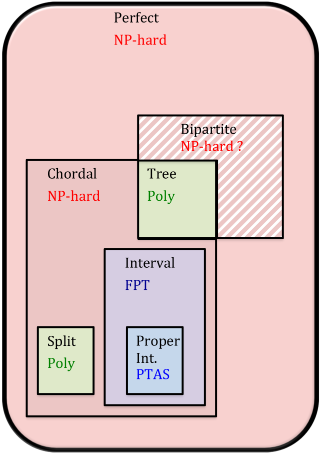
Complexity of k-Sparsest Subgraph:



Open problems and Future Work

Complexity of *k*-Sparsest Subgraph:





Future work/open questions :

- **k-sparsest subgraph:**
  - extend FPT and/or approximation results to Chordal graphs
  - NP-h/Poly on Interval, Proper interval ?
- **k-densest subgraph:**
  - (NP-h/Poly on Interval, Proper interval)
  - FPT/W[1]-hardness on Chordal graphs ?

Merci de votre attention !