On the approximability of the Sum-Max graph partitioning problem

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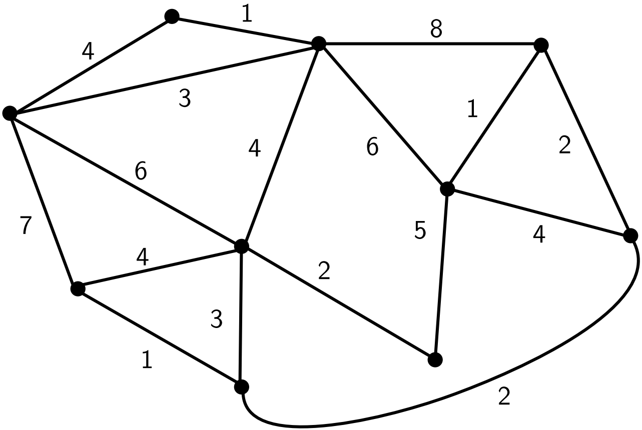

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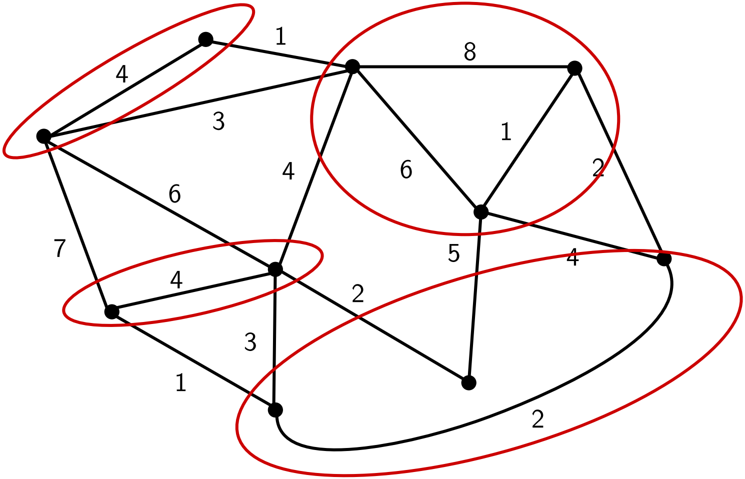

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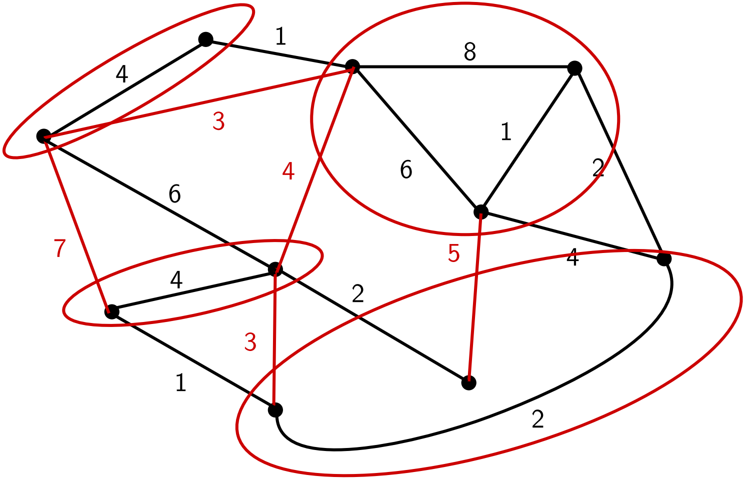

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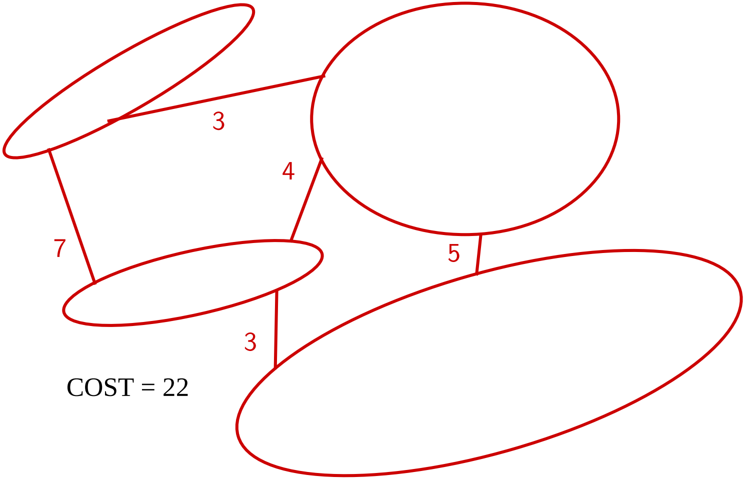

SUM-MAX GRAPH PARTITIONING

Input: a connected graph $G = (V, E)$, $w : E \to \mathbb{N}$, $k \in \mathbb{N}$
Output: a k-partition $(V_1, ..., V_k)$ of V

Goal: minimize $\sum_{\substack{i,j=1 \\ i>j}}^{k} \max_{\substack{u \in V_i \\ v \in V_j}} w(u, v)$

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- simple $\frac{k}{2}$-approximation algorithm
- cannot be approximated with a factor in $O(n^{1-\epsilon})$ (unless $\mathcal{P} = \mathcal{NP}$) (and $\mathcal{NP}$-hardness, $W[1]$-hardness with parameter k)

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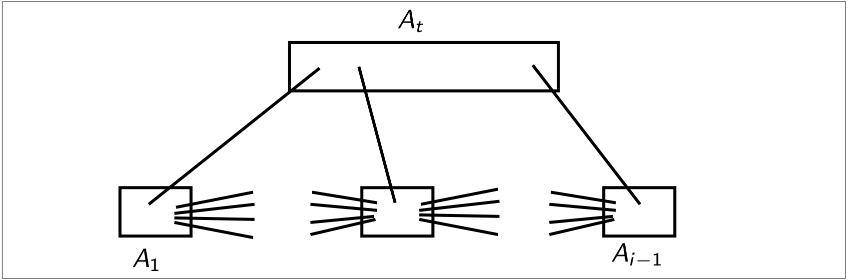

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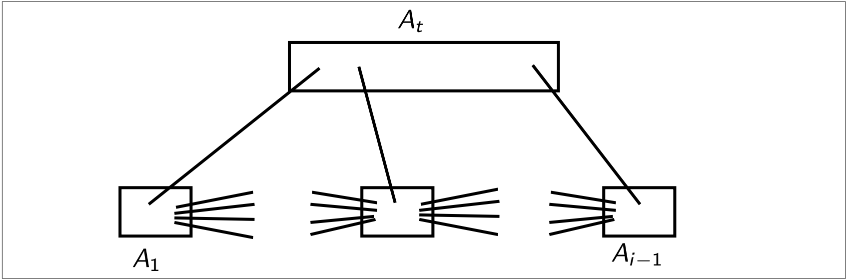

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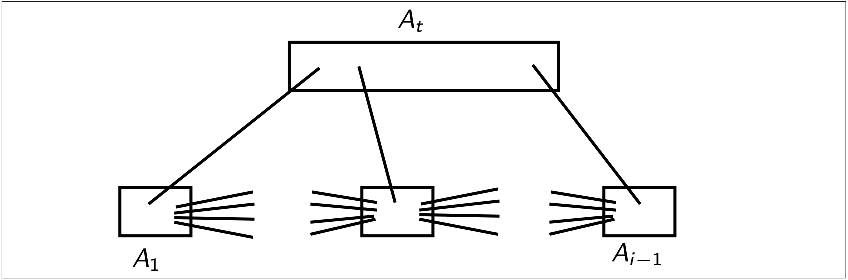

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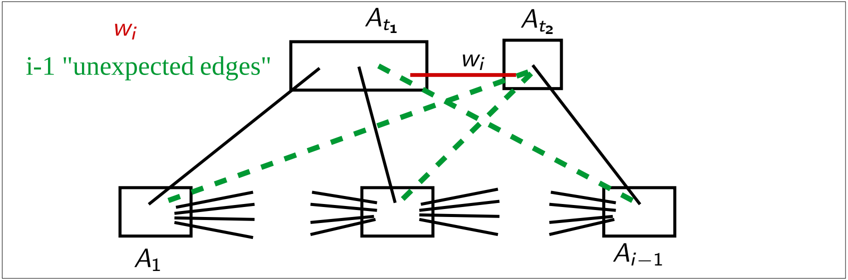

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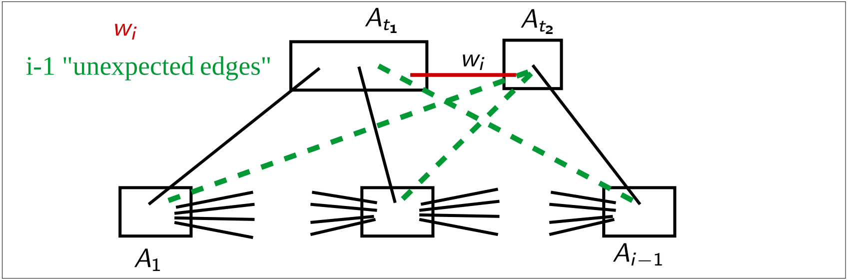

At the end:

$$\text{Solution value} = \sum_{i=1}^{k-1} w_i + \sum \text{unexpected edges}$$

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At each step i: sum of edges of maximum weight outgoing from each cluster is bounded by above by $\sum_{j=1}^{i-1} w_j$

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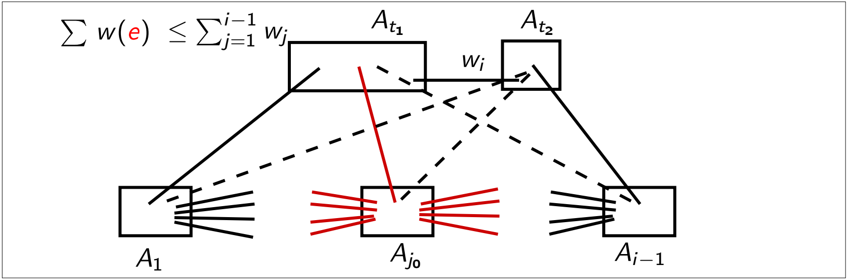

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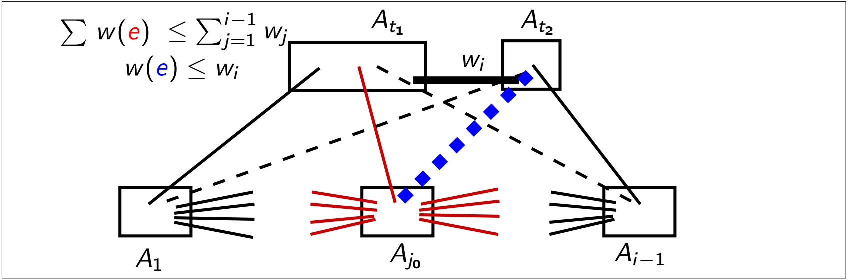

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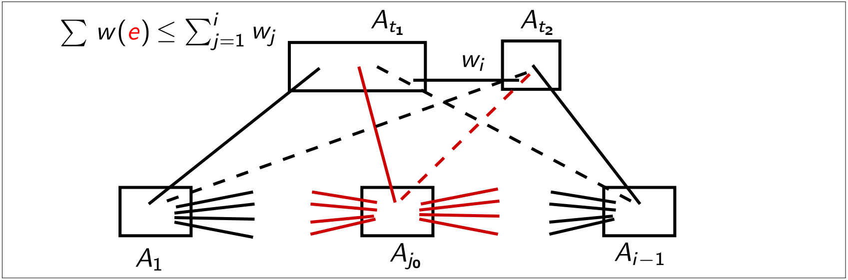

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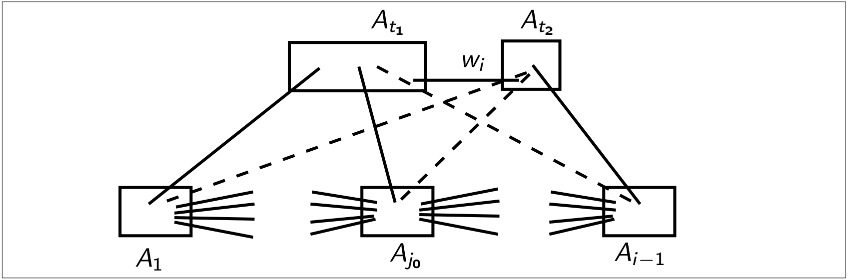

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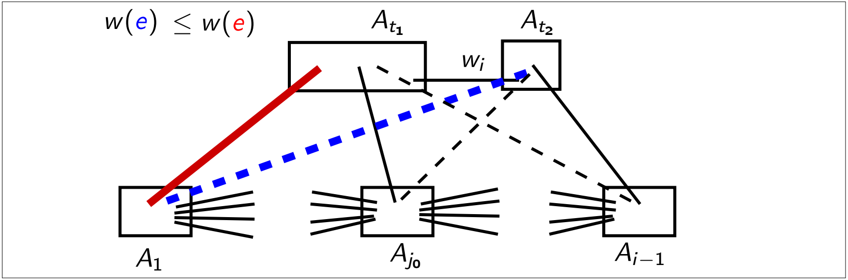

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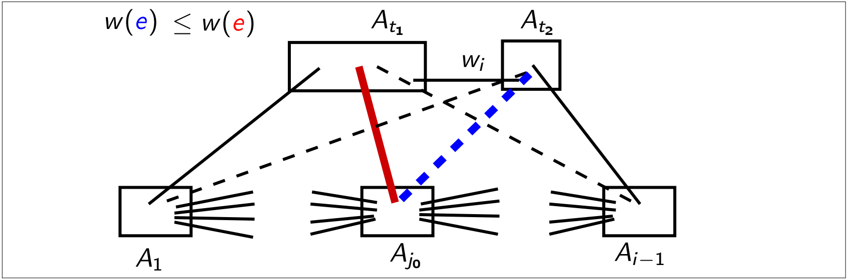

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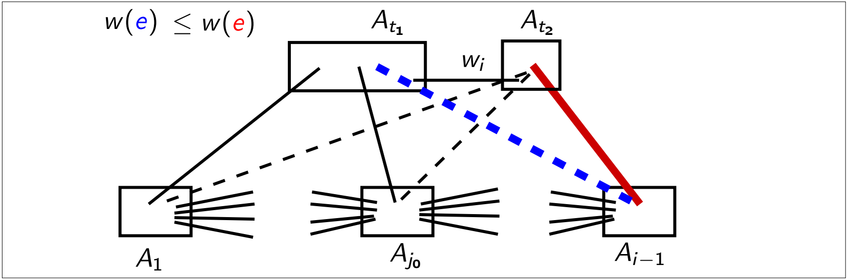

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$$\Rightarrow \mathcal{A} \leq \frac{k}{2} OPT$$

(can be improved using the gap between edge weights)

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$w(e) = 1 \ \forall e \in E$

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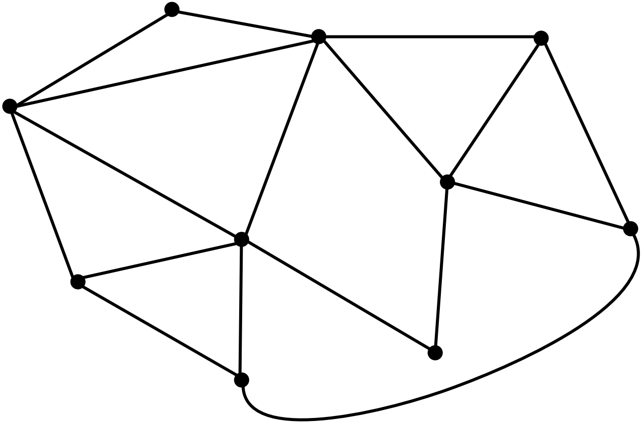

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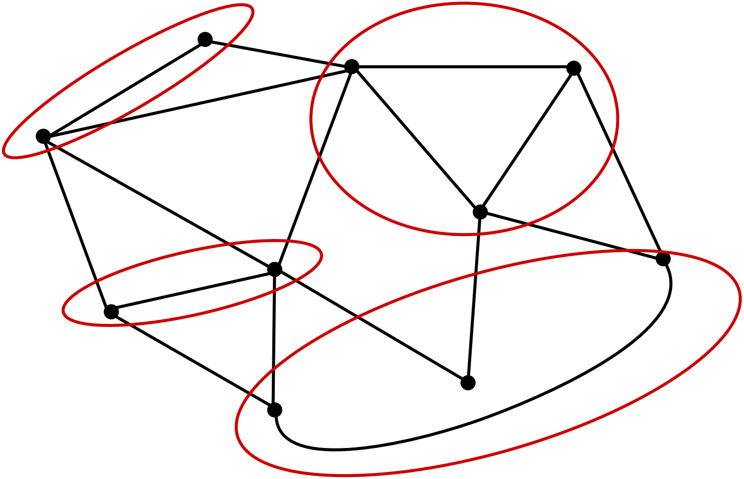

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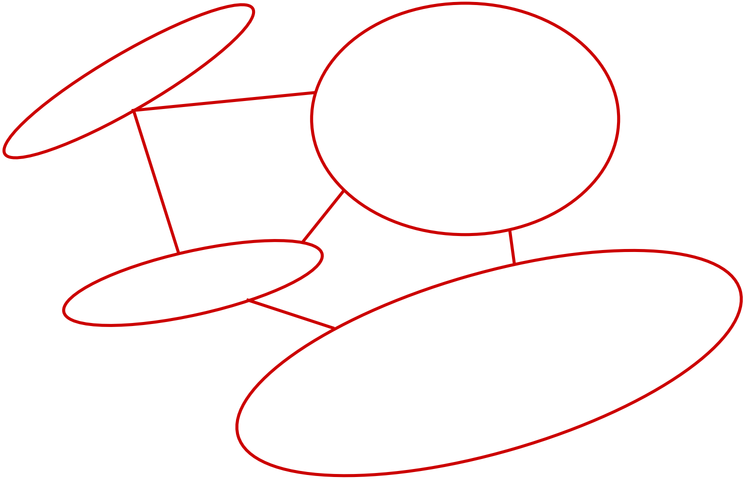

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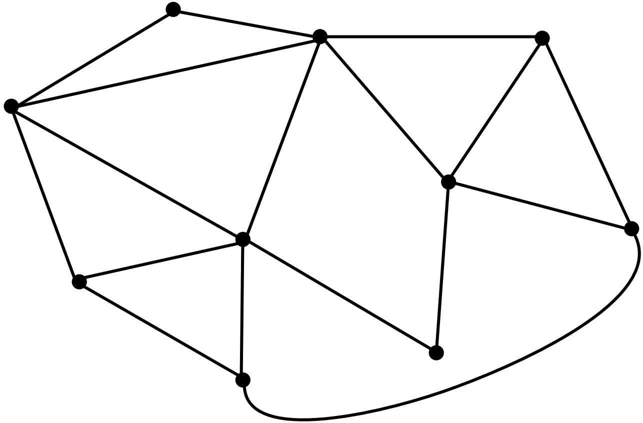

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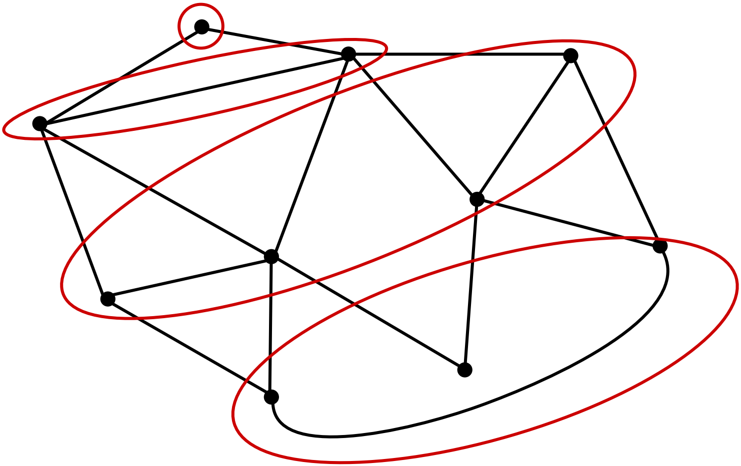

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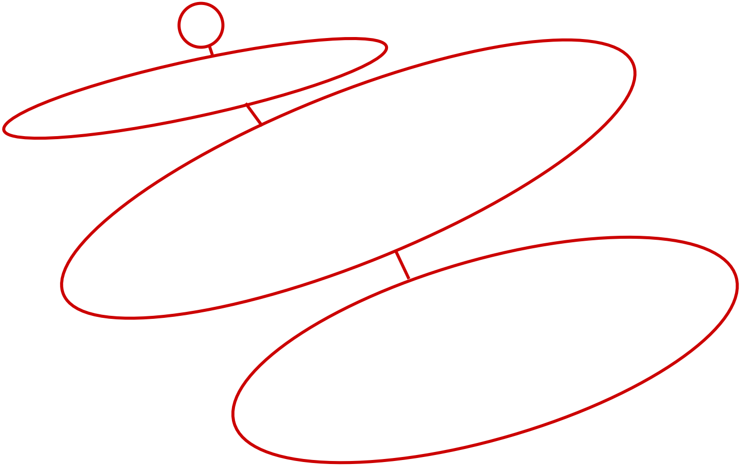

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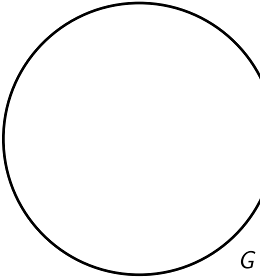

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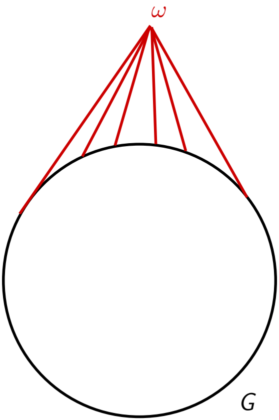

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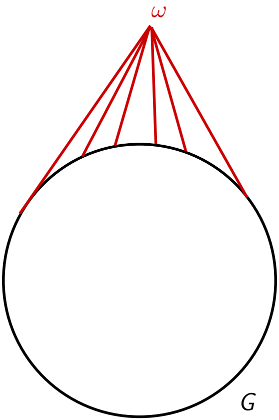

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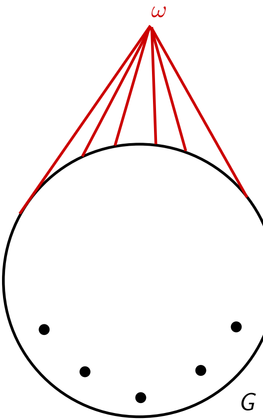

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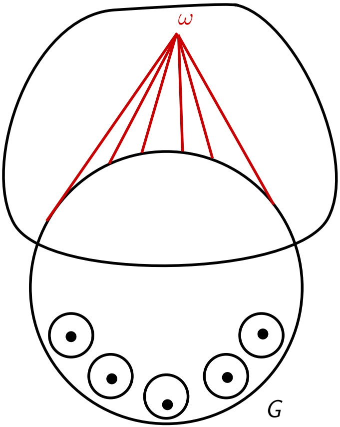

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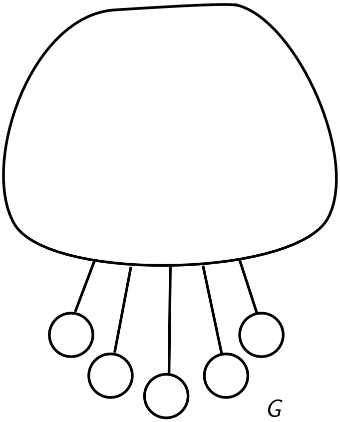

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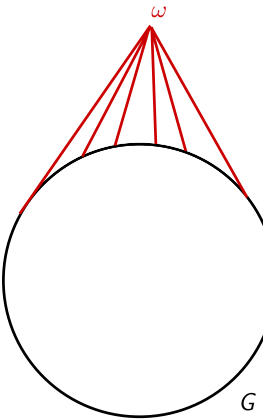

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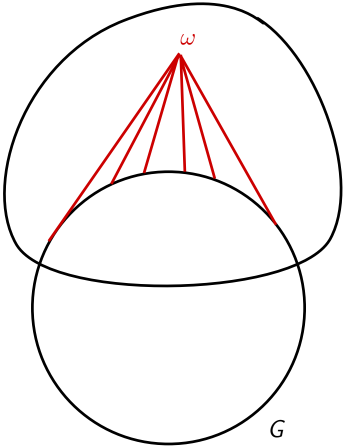

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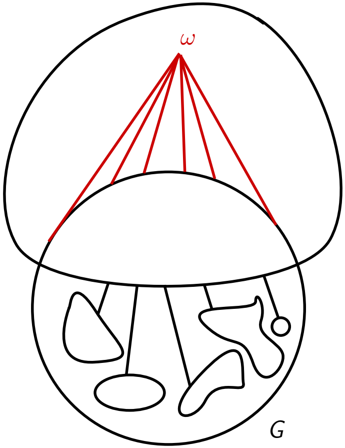

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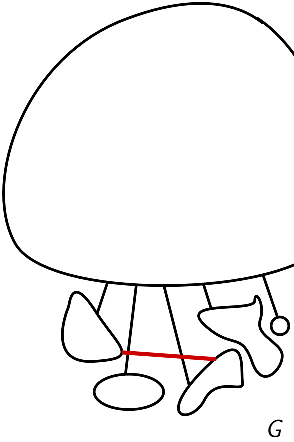

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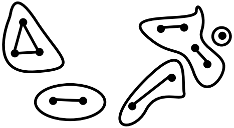

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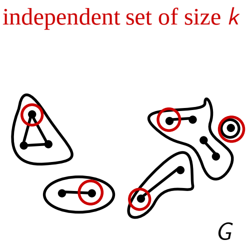

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Reduction from INDEPENDENT SET:

Theorem

SUM-MAX GRAPH PARTITIONING is $\mathcal{NP}$-hard, and even $W[1]$-hard for the parameter k

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SUM-MAX GRAPH PARTITIONING cannot be approximated within $O(n^{1-\epsilon})$

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- if $\alpha(G) \geq k$ then $OPT(G') \leq k$
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  where x = minimum number of edges of any graph with k nodes that does not contain an independent set of size $r.k$

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Given G on n vertices and m edges : $\alpha(G) \geq \frac{n^2}{2m+n}$

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We have:

$$OPT(G') \geq \frac{1}{2r} k$$

$\Rightarrow$ gap preserved :
$O(n^{1-\epsilon})$ non approximable unless $\mathcal{P} = \mathcal{NP}$

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Conclusion, future work

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- no $O(poly(n))$ algorithm, no $O(f(k)poly(n))$ algorithm...
  what about a $O(n^{f(k)})$ algorithm ?
- restricted graph classes :
  - Polynomially solvable in interval graphs (for fixed k)
  - $\mathcal{NP}$-hard in split graphs

Conclusion, future work

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Thank you for your attention!