Kernel lower bound for the k-DOMATIC PARTITION problem

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Contents

1. Kernels, domatic partition
2. hypergraph-2-colorability
3. Transformation to k-DOMATIC PARTITION
4. Conclusion, open question

Kernels, domatic partition

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Kernel

Given a parameterized problem $Q \subseteq \Sigma^* \times \mathbb{N}$, a **kernel** for Q is a **polynomial algorithm** with:

- input: an instance (x, k) of Q
- output: an instance (x', k') of Q

such that:

- $(x, k) \in Q \Leftrightarrow (x', k') \in Q$
- $|x'|, k' \leq f(k)$ for some function f

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Theorem

$Q \in FPT \Leftrightarrow Q$ has a kernel

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Input : a graph $G = (V, E)$
Question : Is there a k-partition of V: $\{V_1, ..., V_k\}$ such that each V_i is a dominating set of G ?

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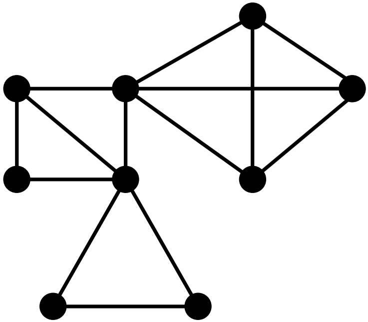

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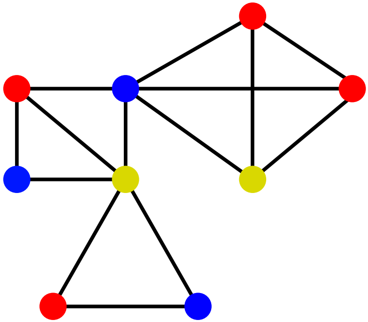

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- FPT when parameterized by *treewidth*(G) (MSO formula)
- 3-DOMATIC PARTITION does **not** admit a polynomial kernel when parameterized by *treewidth*(G) [Bodlaender et al. 2009]
  (unless all coNP problems have a distillation algorithm...)

Hierarchy of parameters

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$\downarrow$

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Our result:

For any fixed $k \geq 3$, k-Domatic Partition does **not** admit a polynomial kernel when parameterized by the size of a **vertex cover** of G
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Sketch of proof:

- **cross-composition** of HYPERGRAPH-2-COLORABILITY to itself
  $\Rightarrow$ no polynomial kernel for HYPERGRAPH-2-COLORABILITY
  (parameterized by the number of vertices)
- **polynomial time and parameter transformation** to k-DOMATIC PARTITION

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Lower bound for HYPERGRAPH-2-COLORABILITY

HYPERGRAPH-2-COLORABILITY

Input : a hypergraph $H = (V, E)$
Question : Is there a bipartition of V into (V_1, V_2) such that each hyperedge has at least one vertex in V_1 and one vertex in V_2 ?
Parameter : $n = |V|$

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Theorem [Bodlaender, Jansen, Kratsch, 2011]

If there exists a **cross-composition** from an $\mathcal{NP}$-complete problem A to a parameterized problem Q, then Q does not admit a polynomial kernel unless $coNP \subseteq NP/Poly$

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A **cross-composition** from a problem $A \subseteq \Sigma^*$ to a parameterized problem $Q \subseteq \Sigma^* \times \mathbb{N}$ is a **polynomial algorithm** with:

- input: a sequence of **equivalent** instances of A: $\{x_1, ..., x_t\}$
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Equivalence relation:

- computable in polynomial time
- partition a set S into less than $\max_{x \in S} |x|^{O(1)}$ classes

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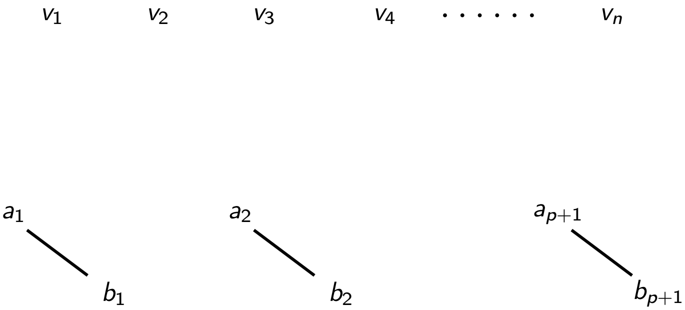

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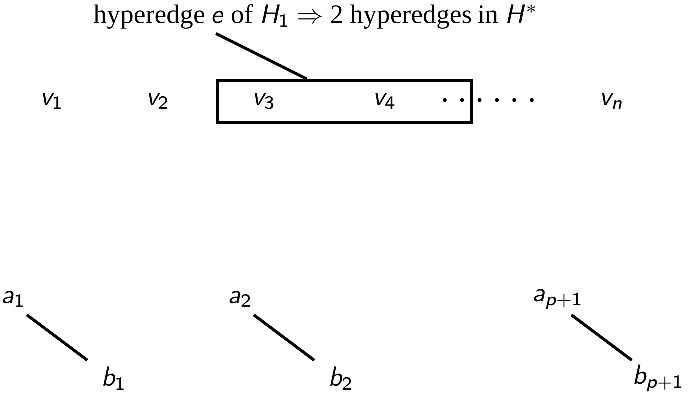

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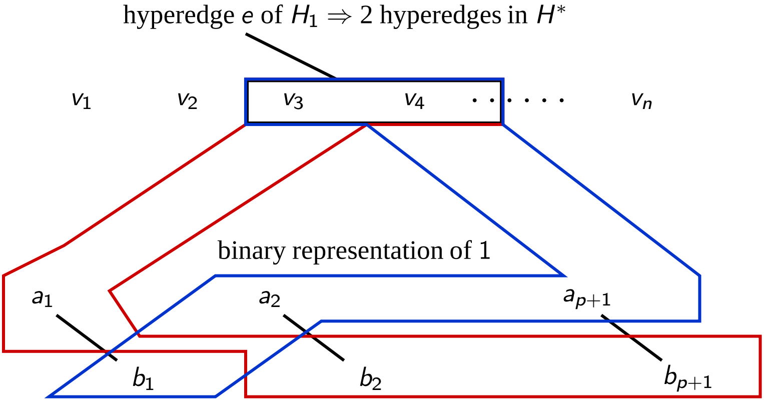

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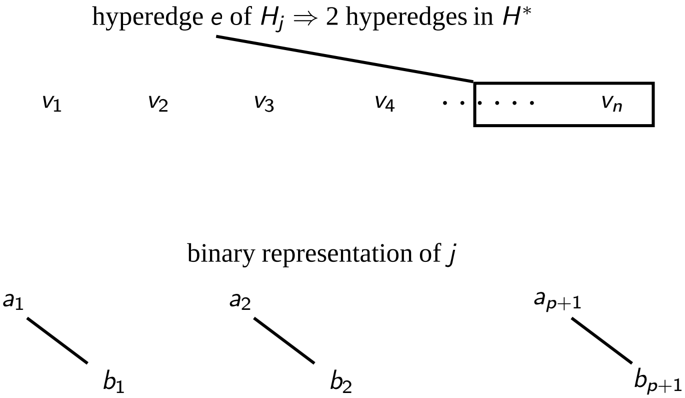

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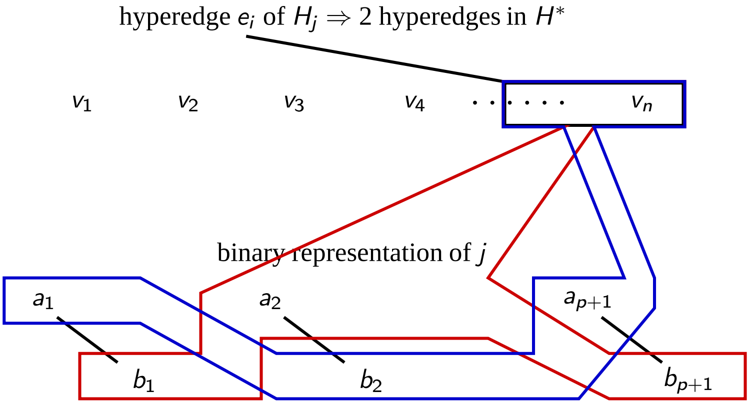

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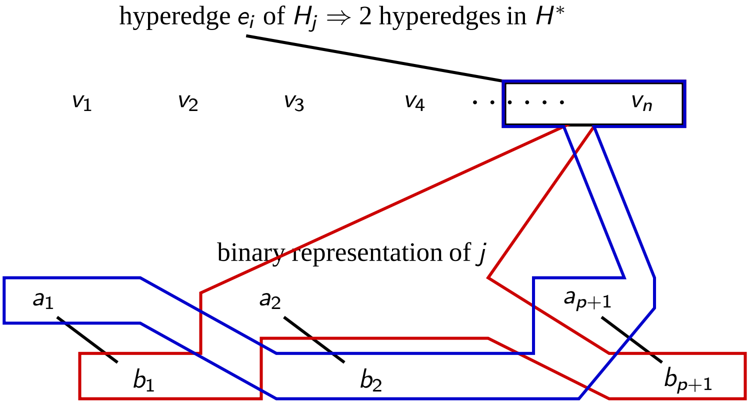

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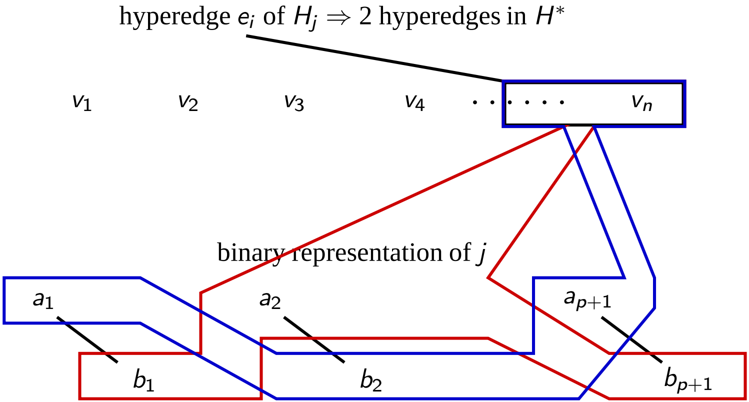

Suppose H_j is a positive instance: there exists a 2-coloring that covers all hyperedges of H_j

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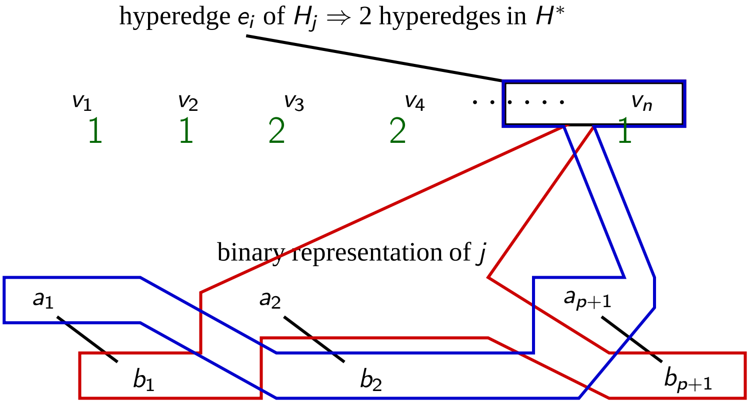

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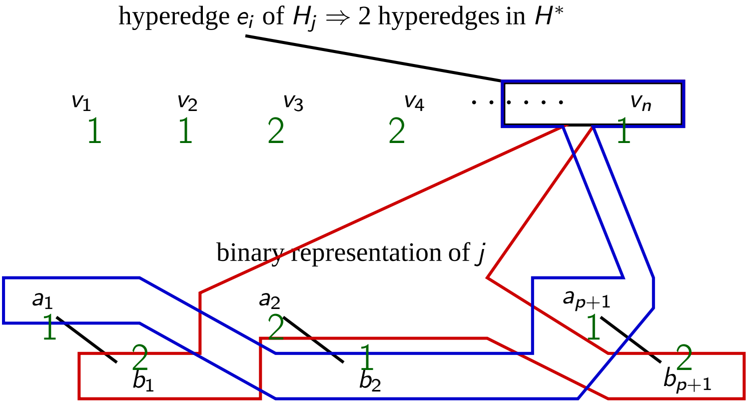

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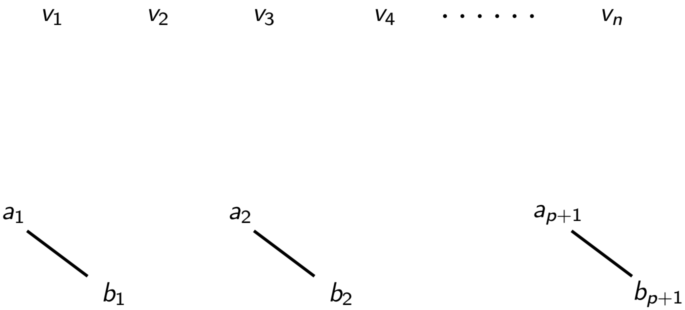

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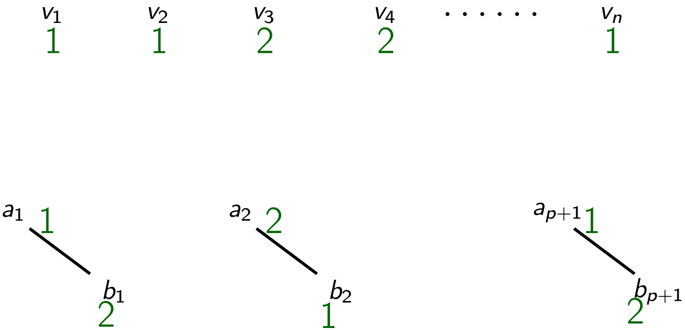

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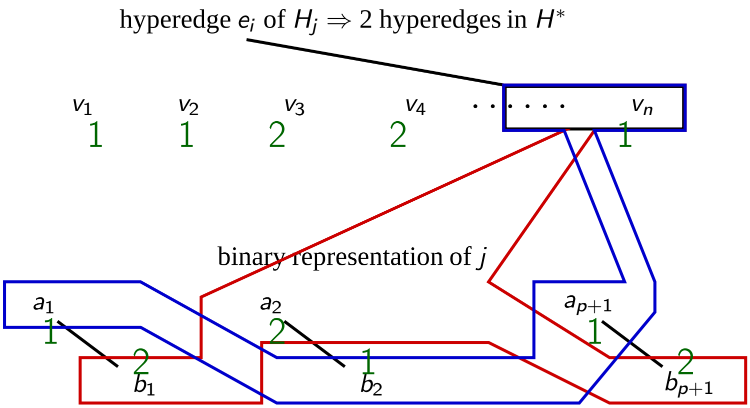

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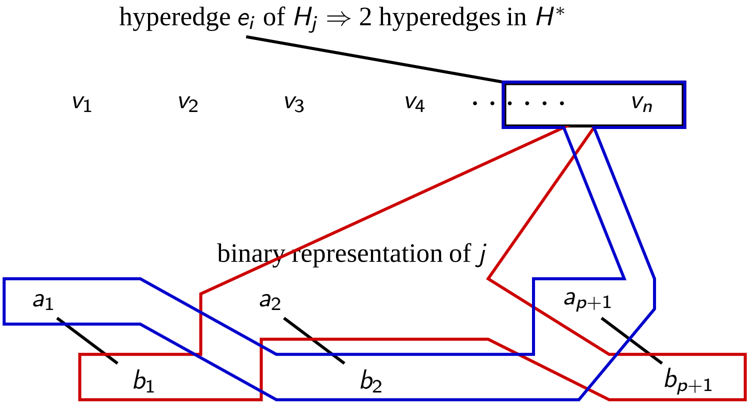

Finally : the number of vertices (parameter) is polynomial in the size of the biggest instance of the sequence + $\log t$

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(proof for $k = 3$, but can be extended for every fixed $k \geq 3$)
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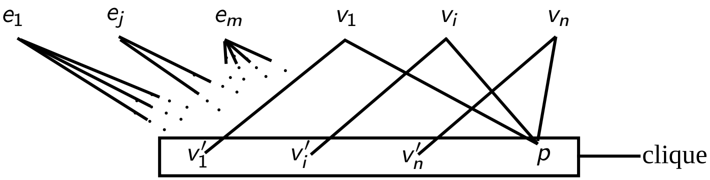

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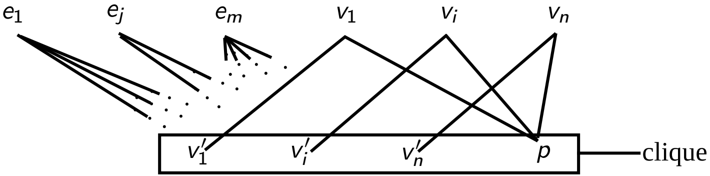
G' has a 3-domatic partition $\Leftrightarrow$ H has a proper 2-coloring.



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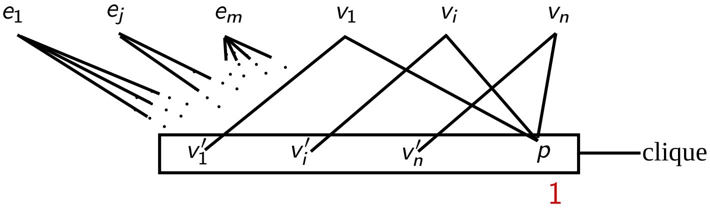

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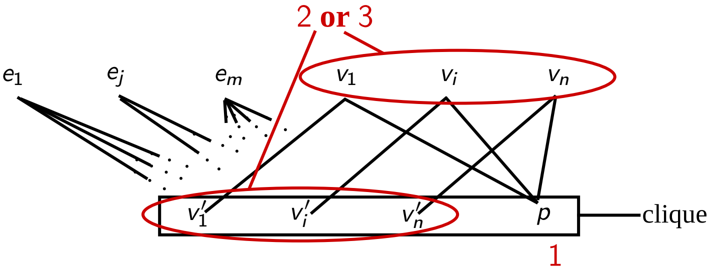

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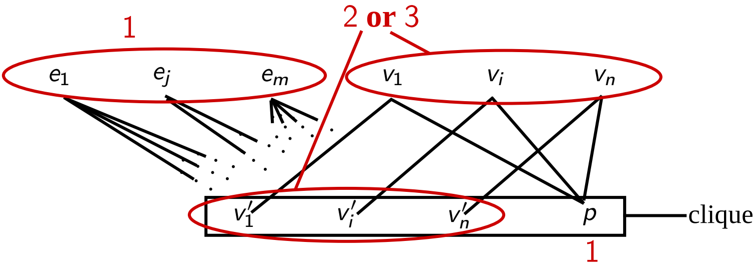
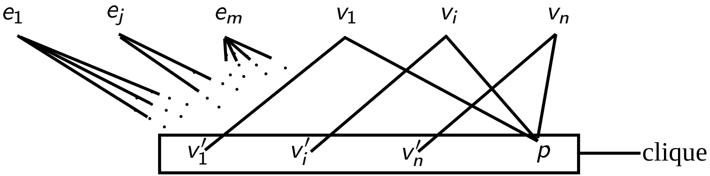


Transformation to k-DOMATIC PARTITION

Finally : the clique is a vertex cover (parameter) of size $n + 1$



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  - distance to other invariants ($CliqueWidth$, $* - width$)
  - here, distance = set of vertices to remove
    - ⋆ set of edges to remove
    - ⋆ set of edges to edit...

Thank you for your attention!