Complexity Dichotomies for a Generic Hypergraph Problem

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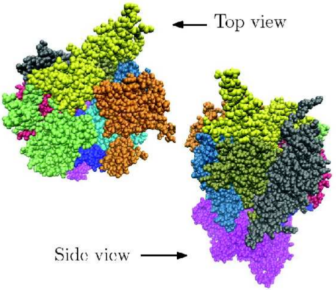
Motivations

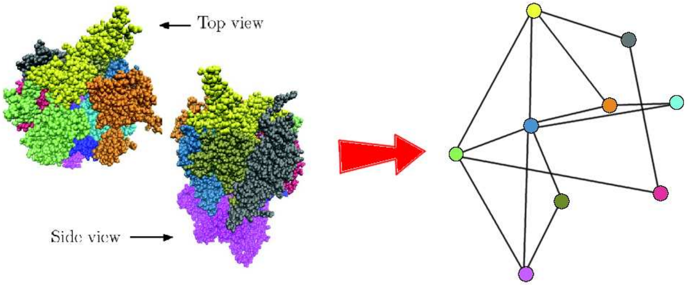
- **Structural biology**: studies the structure of biological macromolecules
  - which sub-units a given complex is made of?
  - how are these sub-units organized?



- Experimental (chemical) methods provide either
  - high resolution (atomic level) of small complexes: X-ray cristallography
  - low resolution of large complexes: mass spectrometry

$\rightarrow$ structure of large complexes?

- Goal: find the **interaction graph**:
  - nodes are the sub-units
  - edge between two sub-units if they are adjacent



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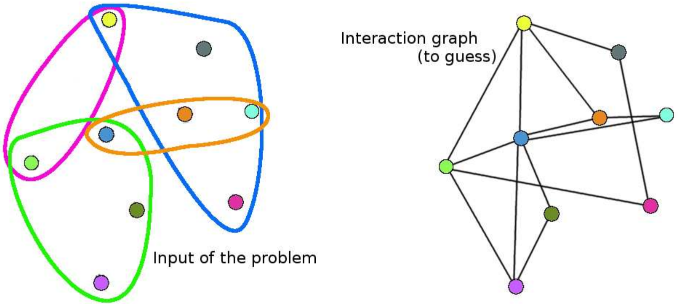

- What is the input of the problem?
  - by modifying the chemical conditions, one can **split the complex into smaller pieces**
  - then, **mass spectrometry** allows us to know:
    - ⋆ the list of all sub-units of the complex
    - ⋆ the sub-units involved in each piece

      → **they form connected subgraphs in the interaction graph**

  ⇒ we obtain a hypergraph

Motivations

Minimum Connectivity Overlay Problem

Input: a hypergraph $H = (V, \mathcal{E})$
Output: a graph $G = (V, E)$ such that:

- for every $S \in \mathcal{E}$, $G[S]$ is connected
- $|E(G)|$ is minimum

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Related work

- studied in different contexts
  - network design
  - users sharing topics of interest (social network)
  - ...
- NP-hard, $O(\log(n))$-approximable, $o(\log(n))$-inapproximable, FPT, ...

Motivations

Minimum ~~Connectivity~~ $\mathcal{F}$ Overlay Problem

Input: a hypergraph $H = (V, \mathcal{E})$
Output: a graph $G = (V, E)$ such that:

- for every $S \in \mathcal{E}$, $G[S]$ is ~~connected~~ <your favourite graph property here>
- $|E(G)|$ is minimum

Related work

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Our objective:

- generalization of the problem to other properties
- for which graph properties the problem is **Polynomial/NP-hard** and **FPT/W[.]-hard**?

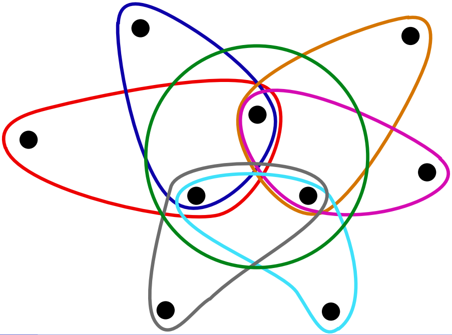
Let $\mathcal{F}$ be a graph family

Minimum $\mathcal{F}$-Overlay

Input: a hypergraph $H = (V, \mathcal{E})$
Output: a graph $G = (V, E)$ such that:

- for every $S \in \mathcal{E}$, $G[S]$ has a spanning subgraph in $\mathcal{F}$
  $\rightarrow$ we say that G ***overlays*** $\mathcal{F}$ on H
- $|E(G)|$ is minimum

Example: $\mathcal{F}$ = the set of all stars



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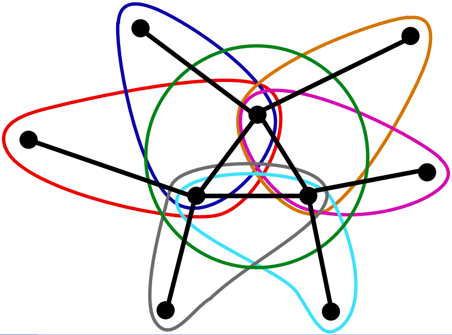

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Some observations:

- if $\mathcal{F}$ is the set of all trees, then we obtain the previous connectivity problem
- G overlays $\mathcal{F}$ on $H \Rightarrow G$ plus any edge overlays $\mathcal{F}$ on H
  $\Rightarrow$ the complete graph on $|V|$ vertices (almost) always overlay $\mathcal{F}$ on H

Our results

- complexity dichotomy: for every $\mathcal{F}$, we can tell whether Minimum $\mathcal{F}$-Overlay is Polynomial or NP-complete
- parameterized algorithms: for almost every $\mathcal{F}$ for which the problem is NP-complete, we can tell whether the problem is FPT or W-hard

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Some obvious polynomial cases:

- if $\mathcal{F}$ contains all edgeless graphs, then the edgeless graph is optimal
- if $\mathcal{F}$ = all cliques, then "a clique on every hyperedge" is optimal

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- if $\mathcal{F} =$ all cliques, then "a clique on every hyperedge" is optimal

These cases are *more or less* the only polynomial ones

Let $\mathcal{F}_p =$ graphs of $\mathcal{F}$ with p vertices

Theorem (easy part)

If, for every $p > 0$, either $\mathcal{F}_p = \emptyset$ or $\mathcal{F}_p = \{K_p\}$ or $\bar{K}_p \in \mathcal{F}_p$,
then **Minimum $\mathcal{F}$-Overlay is polynomial**

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Sketch of the proof (by induction on p)

Def.: $\mathcal{F}^-$ = graphs obtained from $\mathcal{F}_p$ by removing a vertex (all possibilities)

- if $\mathcal{F}^-$ satisfies the statement, we reduce from Minimum $\mathcal{F}^-$-Overlay:

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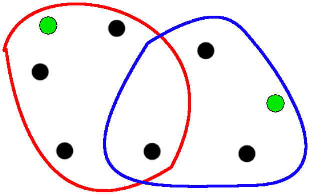
  - G overlays $\mathcal{F}_p$ on the new hypergraph iff it overlays $\mathcal{F}^-$ on the former one



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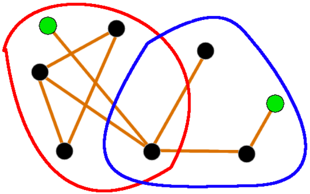

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  - G overlays $\mathcal{F}_p$ on the new hypergraph iff it overlays $\mathcal{F}^-$ on the former one
- what if $\mathcal{F}^-$ is a polynomial case?
  - if $\mathcal{F}^- = \{K_{p-1}\}$, then $\mathcal{F}_p = \{K_p\}$ (impossible)
  - if $\bar{K}_{p-1} \in \mathcal{F}^-$, then $\mathcal{F}_p$ contains a subgraph of the star $K_{1,p}$

Lemma

Minimum $\mathcal{F}_p$**-Overlay** is NP-hard if there is a graph J of order p and two distinct non-edges e_1, e_2 of J such that:

- no subgraph of J (including J itself) is in $\mathcal{F}_p$
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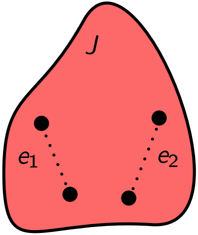

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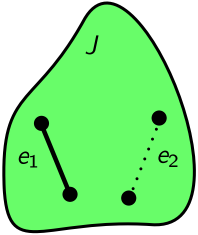

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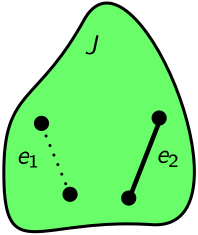

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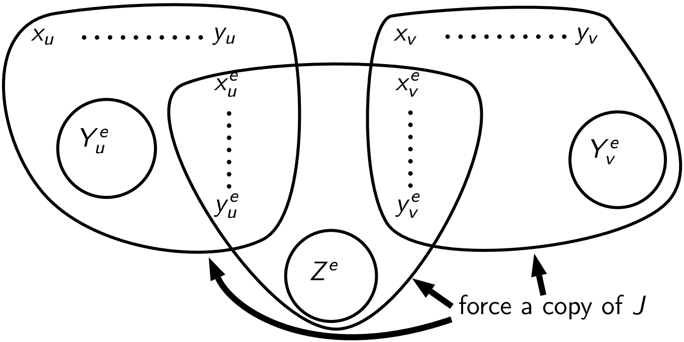

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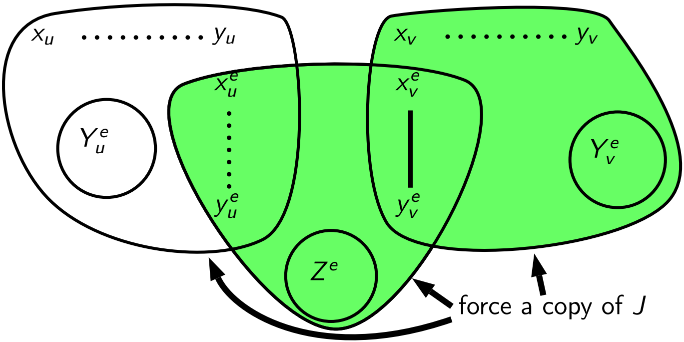

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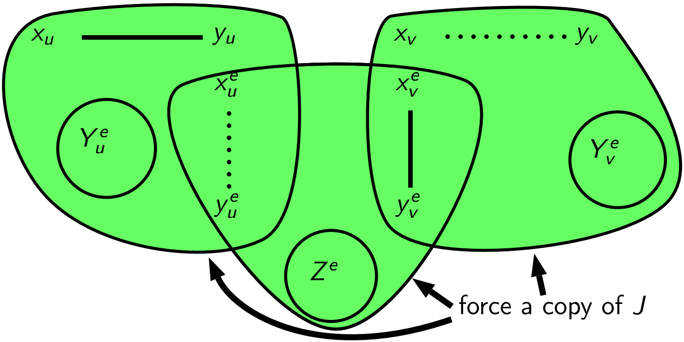

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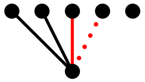
(assume minimality of S)



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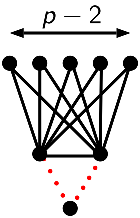

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    - if Q has a vertex of degree 1: ok

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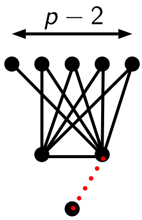

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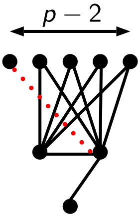

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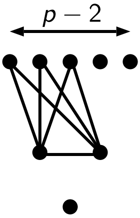

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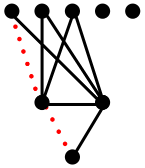

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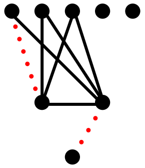

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Parameterized complexity

Minimum $\mathcal{F}$-Overlay is NP-hard for most non-trivial $\mathcal{F}$

- for which $\mathcal{F}$ the problem is FPT or W[1]-hard?

Here: k = "natural parameter" = total number of edges in a solution

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If $\mathcal{F} =$ the set of all trees. Bounded search tree:

- if there is a hyperedge with $\geq k+2$ vertices, answer **"No"**
- otherwise: branch on every possible connected graph for every hyperdege
  $\Rightarrow O^*(2^{k \log(k)})$ algorithm

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Same approach gives:

Theorem

If there is a non-decreasing function $f : \mathbb{N} \rightarrow \mathbb{N}$ with $lim_{n \rightarrow \infty} f(n) = \infty$ such that for all $F \in \mathcal{F}$ we have $|E(F)| \geq f(|V(F)|)$ then **Minimum $\mathcal{F}$-Overlay is FPT**

Parameterized complexity

Theorem

If there is a non-decreasing function $f : \mathbb{N} \to \mathbb{N}$ with $lim_{n\to\infty} f(n) = \infty$ such that for all $F \in \mathcal{F}$ we have $|E(F)| \geq f(|V(F)|)$ then **Minimum $\mathcal{F}$-Overlay is FPT**

Examples of $\mathcal{F}$ satisfying the statement:

- whenever $\mathcal{F}$ is finite
- $\mathcal{F} =$ all stars
- $\mathcal{F} =$ hamiltonian graphs
- $\mathcal{F} =$ graphs of minimum degree d
- $\mathcal{F} =$ c-connected graphs
- ...

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- graphs having an arbitrary number of isolated vertices
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Theorem

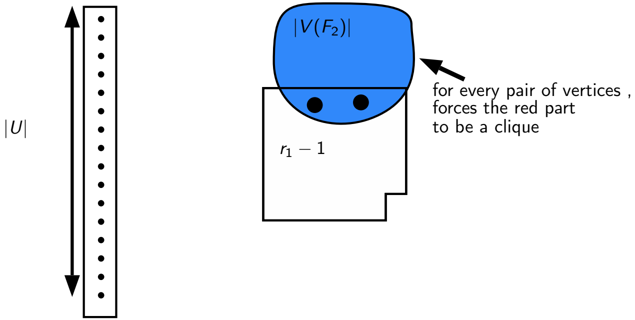
Let $\mathcal{F}$ be a loose family of graphs.
If $\bar{K_p} \in \mathcal{F}$ for some p, then Minimum $\mathcal{F}$-Overlay is FPT, otherwise, it is W[1]-hard

W[1]-hardness

Let $U, \mathcal{S}, k$ be a **Hitting Set** instance (U=Universe, $\mathcal{S}$=subsets of U, $k \in \mathbb{N}$)

F_1 = graph of $\mathcal{F}$ with min. number of non-isolated vertices r_1

F_2 = graph of $\mathcal{F}$ with min. number of edges



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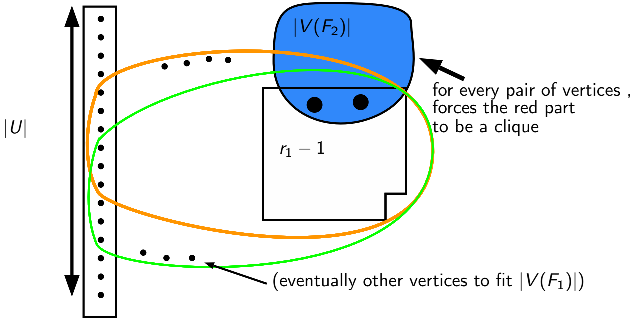

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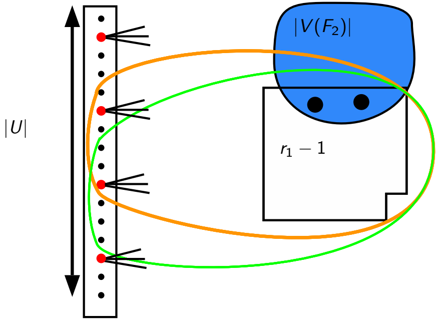

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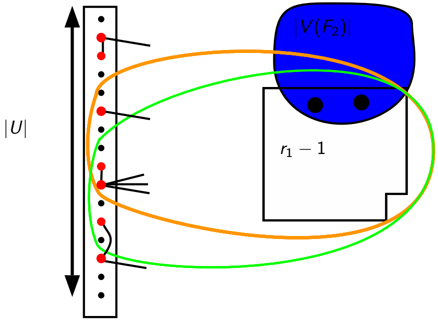

$$k' = \binom{|V(F_1)|-1}{2}|E(F_2)| + k\delta(F_1)$$

- Conversely: right part must be a clique $\Rightarrow$ left part covers $\leq k\delta(F_1)$ edges.
- The non-isolated vertices of the left part is a hitting set
  But: no guarantee that it is an independent set (e.g.: F_1 disconnected)
  $\Rightarrow$ What is the maximum number of (non-isolated) vertices that can cover $k\delta(F_1)$ edges?

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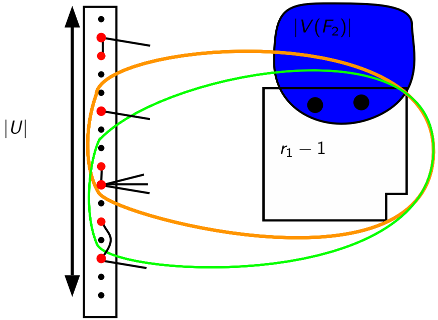

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Theorem [Chen, Lin, FOCS 2016]

Approximating Hitting Set to any constant is W[1]-hard

$\Rightarrow$ reduce from $\text{Gap}_{2\delta(F_1)}$ Hitting Set

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Open problems, further research

Parameterized algorithms

- what about $\mathcal{F}$ which are not loose, but does not fall into the FPT case?
  - "almost loose": for all $F \in \mathcal{F}$, $F + \bar{K}_{g(i)} \in \mathcal{F}$ $\forall i$
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  - what if $g(i) = 2^i$?

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Variants of the problem

- require that for every hyperedge $S \in \mathcal{E}$, $G[S]$ is **isomorphic** to some $F \in \mathcal{F}$
  $\Rightarrow$ forbids additional edges

  now, testing satisfiability is no longer polynomial
  NP-hard even if $\mathcal{F} = \{P_3\}$
  complexity dichotomy?

- add some constraints on the output graph:
  - $\Delta(G) \leq d$
  - bounded treewidth?
  - already some work with "planarity" constraint (hypergraph drawing)

Voilà !
Questions ?