Projectively Invariant Intersection Detections for Solid Modeling

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An intersection detection method for solid modeling which is invariant under projective transformations is presented. We redefine the fundamental geometric figures necessary to describe solid models and their dual figures in a homogeneous coordinate representation. Then we derive conditions, which are projectively invariant, for intersections between these primitives. We will show that a geometric processor based on the 4×4 determinant method is applicable to a wide range of problems with little modification. This method has applications in intersection detections of rational parametric curves and surfaces and hidden-line/surface removal algorithms.

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1. INTRODUCTION

Homogeneous coordinates have come to be used extensively in solid modeling and computer graphics. For example, homogeneous coordinates are useful in perspective transformations. Homogeneous coordinates represent control points for rational parametric curves and surfaces. Also, determinants consisting of homogeneous coordinates are used in intersection detections [Yamaguchi 1985; Yamaguchi and Niizeki 1990; Yamaguchi et al. 1993].

There are two main types of geometric computations performed in modeling applications. One is geometric transformation, and the other is intersection detection. Scaling, translations, rotations, reflections, and perspective transformations are examples of projective transformations. By using homo-

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geneous coordinates, projective transformations may be dealt with simply, generally, and in a unified way as the multiplication of 4×4 matrices.

The 4×4 Determinant Method provides a structured method of detecting and computing geometric intersections using homogeneous coordinates throughout the modeling process [Yamaguchi 1985; Yamaguchi and Niizeki 1990; Yamaguchi et al. 1993]. Intersections are detected by checking the signs of 4×4 determinants consisting of homogeneous coordinate vectors of points or planes. A great part of the geometric computation in applications such as Boolean set operations, hidden-line and surface elimination, and ray tracing can be conducted using the 4×4 determinant as its sole computation primitive.

If we use 4×4 matrices to perform geometric transformations and 4×4 determinants to perform geometric intersection computations, it becomes possible to use homogeneous coordinates throughout the modeling process without returning to nonhomogeneous coordinates. This approach enables us to develop a geometric processing package which handles all geometric calculations for solid modeling. Besides the theoretical significance of this unified approach, this approach has also contributed in enhancing computational efficiency and numerical stability [Yamaguchi and Niizeki 1990; Yamaguchi et al. 1993]. Since the geometric algorithms and computations are extremely simple, this approach is also suited for hardware implementation. The POLYGON ENGINE is a general-purpose geometric processor based on this approach [Yamaguchi et al. 1988].

In spite of the uniformity of the representation and operations enabled by homogeneous coordinates, they have not been fully exploited. One aspect of the homogeneous coordinate representation which has not been thoroughly investigated is the projective invariance of intersection detections. Traditional methods of intersection detection do not always operate properly between figures obtained as a result of projective transformation (for example, perspective transformation). There are cases where test results show that two figures do not intersect after a projective transformation although they originally intersect before the transformation. Take the two line segments P_0P_1 and P_aP_b in Figure 1(a) as an example. We will examine a two-dimensional example for simplicity, but we can easily see that the same problem also arises in three or more dimensions. The line segments have the following Euclidean coordinates:

$$P_0 = (0,0), \qquad P_1 = (1,1), \qquad P_a = (0,1), \qquad P_b = (1,0).$$

Now, consider the case where we must conduct a projective transformation on these line segments. In these circumstances, it is very convenient for us to use homogeneous coordinates and multiply a single projective transformation matrix. First, we represent the end points of the line segments using homogeneous coordinates by adding a scale factor as the third coordinate as follows:

$$V_0 = [0 \quad 0 \quad 1], \qquad V_1 = [1 \quad 1 \quad 1], \qquad V_a = [0 \quad 1 \quad 1], \qquad V_b = [1 \quad 0 \quad 1].$$

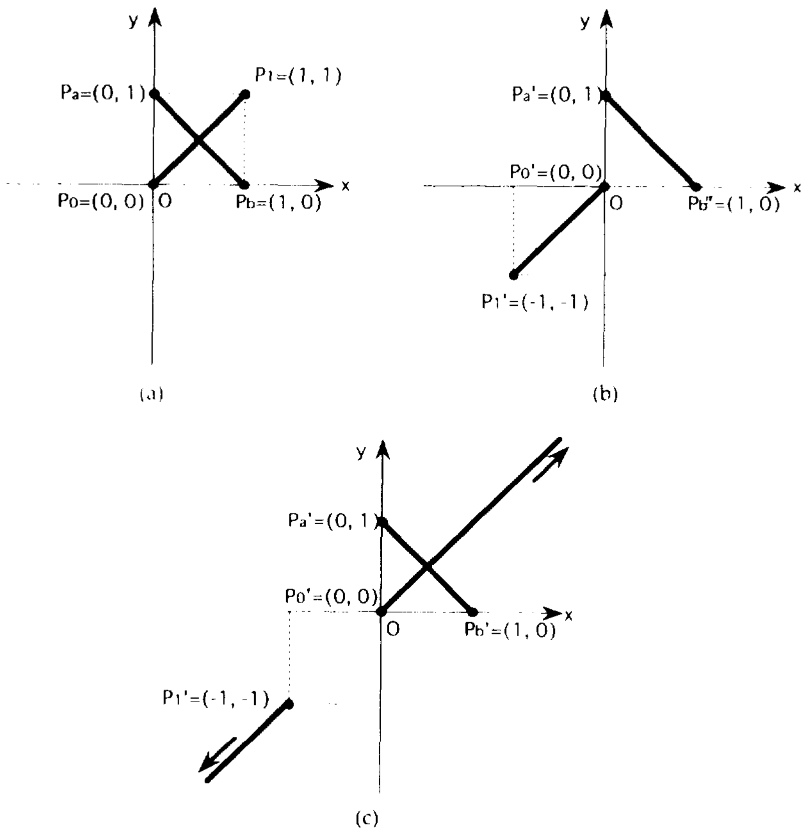


Fig. 1. Two-dimensional case of a line intersection before and after a projective transformation.

We can then postmultiply an arbitrary projective transformation matrix, for example,

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{bmatrix}.$$

We obtain the following homogeneous coordinates:

$$V_0' = [0 \quad 0 \quad 3], \qquad V_1' = [1 \quad 1 \quad -1],$$
$$V_a' = [0 \quad 1 \quad 1], \qquad V_b' = [1 \quad 0 \quad 1],$$

which correspond to the Euclidean points in Fig. 1(b):

$$P_0' = (0,0), \qquad P_1' = (1,1), \qquad P_a' = (0,1), \qquad P_b' = (1,0).$$

We find that the line segments $P_0'P_1'$ and $P_a'P_b'$ obtained from the projective transformation of the original points do not intersect. This is due to the effect of "wrapping around infinity," which many papers point out. (See for example Blinn [1993] and Blinn and Newell [1978].) The projective transformation of an ordinary line segment does not always produce an ordinary "internal" line segment, but sometimes an "external" line segment (which is the case in Figure 1(c)), as well as many other peculiar types of line segments. A means of consistent intersection detection based on homogeneous coordinates is necessary for these types of line segments. These projectively invariant intersection detection tests would allow us to test intersection before or after an arbitrary projective transformation. This would guarantee that we can conduct intersection tests (such as tests for clipping and hidden-line/surface removal) at any convenient part of a viewing pipeline for display of three-dimensional shapes.

Similar problems can be seen in intersection detections concerning rational parametric curve segments and surface patches when there are control points with weights having mixed signs. These curves and surfaces are considered not to have the convex-hull property. There is no consistent method of detecting their intersection using subdivision techniques. This makes rational curves and surfaces inconvenient compared to the nonrational representation.

Another important aspect of the homogeneous coordinate representation which has not been investigated quite fully is the duality between points and planes. Duality in geometric intersection problems has always been appealing to researchers [Arokiasamy 1989; Chazelle et al. 1983]. Even so, there has not been a thorough treatment of this subject in literature, especially in the context of three-dimensional geometric modeling. For a method of geometric computation based on the homogeneous coordinate representation to be complete, the principle of duality must not be disregarded.

The purpose of this article is to present a projectively invariant method of intersection detection for solid modeling and computer graphics applications. We will redefine the basic concepts and primitives used in geometric modeling using homogeneous coordinates and investigate the properties of these primitive figures. We will define the duals of these primitives and investigate their properties, also. Then we will derive methods, which are projectively invariant, for detecting intersections between the primitives. These intersection tests will allow shorter and more orthogonal code in geometric processing packages.

These new concepts and methods will enable us to detect intersections correctly before and after an arbitrary projective transformation. They will enable us to detect intersections between primitives containing homogeneous coordinates with weights having arbitrary signs. We will also obtain a complete system of projectively invariant geometric intersection tests between primitives defined by planes which will be shown to be the dual of the system for points. Since our main purpose is to discuss the theoretical foundations for a geometric intersection-testing method, we will deliberately leave out discussions concerning implementation issues such as algorithmic efficiency and numerical accuracy. These topics have been discussed in detail in Yamaguchi [1985; 1987] and Yamaguchi et al. [1988].

Another important purpose of this article is to show that a single geometric processing package or hardware processor can be applied to a wider range of problems with little modification. In the course of designing a geometric hardware processor based on the 4×4 determinant method, we have found that the package or processor is capable of processing the homogeneous and dual primitives defined in this article as well as the conventional Euclidean ones. The new concepts and algorithms presented here will enable us to make the most of the geometric processor. A few examples of the applications of the new intersection detection methods will also be presented to show its usefulness.

2. DEFINITIONS OF HOMOGENEOUS FIGURES

2.1 Homogeneous Coordinates

A point in three-dimensional space is represented in homogeneous coordinates by means of a four component nonzero row vector, usually written as $\mathbf{V} = [X \ Y \ Z \ w]$. Any nonzero multiple of this vector $\lambda\mathbf{V} = [\lambda X \ \lambda Y \ \lambda Z \ \lambda w]$ ($\lambda \neq 0$) represents the same point. Note that throughout this article, the two homogeneous coordinate vectors $[X \ Y \ Z \ w]$ and $[-X \ -Y \ -Z \ -w]$ represent the same point, in contrast to the approach taken by Stolfi [1987]. To obtain the corresponding Cartesian coordinates of this point, we divide each component by w, unless $w = 0$. The first three components obtained by this division, $(X/w, \ Y/w, \ Z/w)$ are the conventional three-dimensional coordinates of this point. If $w = 0$, the homogeneous coordinate vector represents a point at infinity in the direction of the three-dimensional vector $[X \ Y \ Z]$, which is not representable by ordinary Cartesian coordinates. The set of points with $w = 0$ is called the plane at infinity. The w's are called the weights (or scale factors) of the homogeneous coordinate vectors. Throughout this article, the weights may have any sign (i.e., either positive, zero, or negative). Since we consider any nonzero multiple of a homogeneous coordinate vector to represent the same point, we will refer to the point represented by the homogeneous coordinate vector $\mathbf{V}$ as "the point $\mathbf{V}$" when this does not cause any confusion.

2.2 Homogeneous Line Segments

Now we take the basic primitives in modeling and redefine them as new concepts, using homogeneous coordinates.

An ordinary line segment in three-dimensional Euclidean space may be expressed as a linear combination of the position vectors of two distinct points. Here, we consider a linear combination of two homogeneous coordinate vectors which represent two distinct points.

$$\begin{aligned} \mathbf{V} &= \xi_0\mathbf{V}_0 + \xi_1\mathbf{V}_1, \\ &\text{where } (\xi_0, \xi_1) \neq (0,0) \\ &\text{and } (\xi_0, \xi_1 \geq 0 \text{ or } \xi_0, \xi_1 \leq 0). \end{aligned} \tag{1}$$

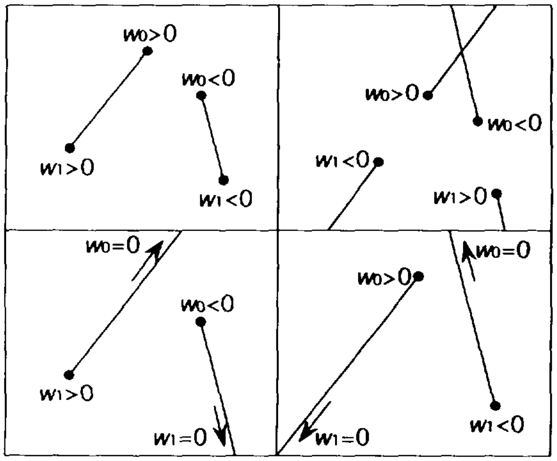


Fig. 2. Examples of homogeneous line segments.

It is easily shown that when the weights of the homogeneous coordinate vectors of the two end points have the same sign (positive or negative), the set of points expressed by this equation represents an ordinary (internal) line segment. But when one of the weights is zero, this equation represents a half-line. When the weights of the two vectors have different signs, this set represents an "external" line segment. The set of points represented by this equation will be called the *Homogeneous Line Segment* defined by the homogeneous coordinate vectors $\mathbf{V}_0$ and $\mathbf{V}_1$, or more briefly, the homogeneous line segment $\mathbf{V}_0\mathbf{V}_1$. Note that a homogeneous line segment is not defined by two points, but two homogeneous coordinate vectors. The projective transformation of a homogeneous line segment produces another homogeneous line segment. Figure 2 shows various types of homogeneous line segments with differently signed weights. The case where both weights are zero is not shown because it lies on the plane at infinity and cannot be drawn. It is possible to represent a homogeneous line segment using only one parameter, but here we use the two parameter representation for two reasons. The first reason is that this *homogeneous* parameter representation is much more useful in the derivation of the projectively invariant intersection conditions presented in the following sections. The second reason is that, in a geometric computing environment where all coordinates are represented in homogeneous coordinates, it is far more natural for parametric representations to take a homogeneous form. The advantages of such a representation relates closely to the advantages of the homogeneous parameterization of a rational Bézier curve in Patterson [1985]. We obtain the ordinary one-parameter representation of a line segment by dividing both sides of (1) by $\xi_0 + \xi_1 (\neq 0)$ and by eliminating one of the parameters. We may obtain a homogeneous line segment from an ordinary line segment by performing a projective transformation on it.

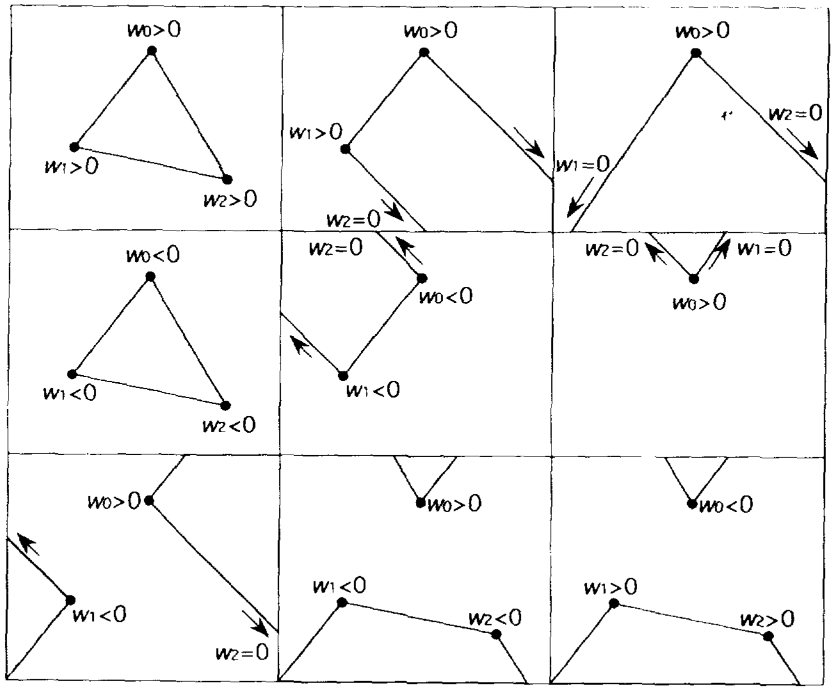


Fig. 3. Examples of homogeneous triangles.

Homogeneous line segments have been defined and studied by Blinn and Newell [1978]. The formulation here is different from that in Blinn and Newell, but the concept explained here is the same.

2.3 Homogeneous Triangles

Next, we consider a linear combination of homogeneous coordinate vectors which represent three noncollinear points.

$$\begin{aligned}\mathbf{V} &= \xi_0\mathbf{V}_0 + \xi_1\mathbf{V}_1 + \xi_2\mathbf{V}_2,\\ &\text{where } (\xi_0, \xi_1, \xi_2) \neq (0,0,0) \\ &\text{and } (\xi_0, \xi_1, \xi_2 \geq 0 \text{ or } \xi_0, \xi_1, \xi_2 \leq 0).\end{aligned} \tag{2}$$

This set of points will be called the *Homogeneous Triangle* $\mathbf{V}_0\mathbf{V}_1\mathbf{V}_2$. It is easily shown that when the weights of the homogeneous coordinate vectors of the three points have the same sign (positive or negative), the set of points expressed by this equation represents an ordinary triangle. Figure 3 shows homogeneous triangles with weights having different signs. Homogeneous triangles with all three weights equal to zero are not shown, because they lie entirely on the plane at infinity. The projective transformation of a homogeneous triangle produces another homogeneous triangle. We may obtain a homogeneous triangle from an ordinary triangle by performing a projective

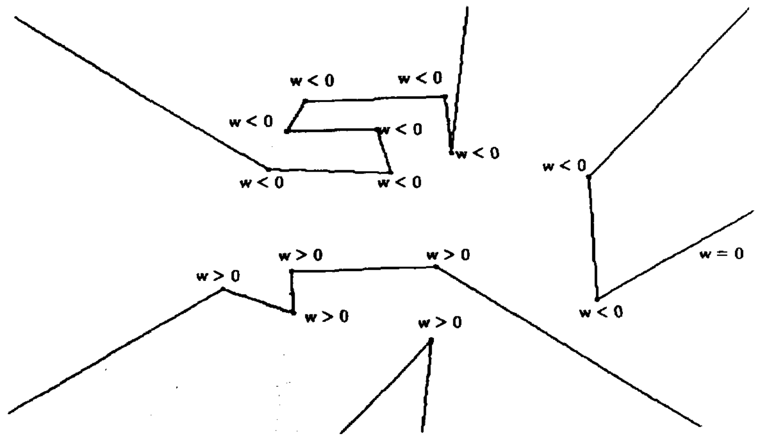


Fig. 4. A homogeneous polygon.

transformation on it. The three sides of a homogeneous triangle are homogeneous line segments.

2.4 Homogeneous Polygons

A *Homogeneous Polygon* is defined as a cyclic sequence of homogeneous coordinate vectors representing coplanar points. We assume that successive homogeneous coordinate vectors in this cyclic sequence define a closed loop of homogeneous line segments which do not intersect each other, except for the end points of successive line segments. Then this loop divides the plane into two parts. Since a plane in homogeneous coordinate space is topologically equivalent to a projective plane, one part is homeomorphic to a disc, while the other is homeomorphic to a Möbius strip. We will call the part of the plane which is homeomorphic to a disc and its boundary a *Homogeneous Polygon*. Figure 4 shows an example of a homogeneous polygon. Since a homogeneous polygon is homeomorphic to a disc, it is orientable and can be triangulated. When the weights of the homogeneous coordinate vectors all have the same sign (positive or negative), a homogeneous polygon is an ordinary polygon. The projective transformation of a homogeneous polygon produces another homogeneous polygon. We may obtain a homogeneous polygon by projectively transforming an ordinary polygon.

2.5 Homogeneous Tetrahedra

Next, we consider a linear combination of homogeneous coordinate vectors which represent four points. The four points must be noncoplanar, and each triple of points must be noncollinear.

$$\begin{aligned} \mathbf{V} = {} & \xi_0 \mathbf{V}_0 + \xi_1 \mathbf{V}_1 + \xi_2 \mathbf{V}_2 + \xi_3 \mathbf{V}_3, \\ & \text{where } (\xi_0, \xi_1, \xi_2, \xi_3) \neq (0,0,0,0) \\ & \text{and } (\xi_0, \xi_1, \xi_2, \xi_3 \geq 0 \text{ or } \xi_0, \xi_1, \xi_2, \xi_2 \leq 0). \end{aligned} \tag{3}$$

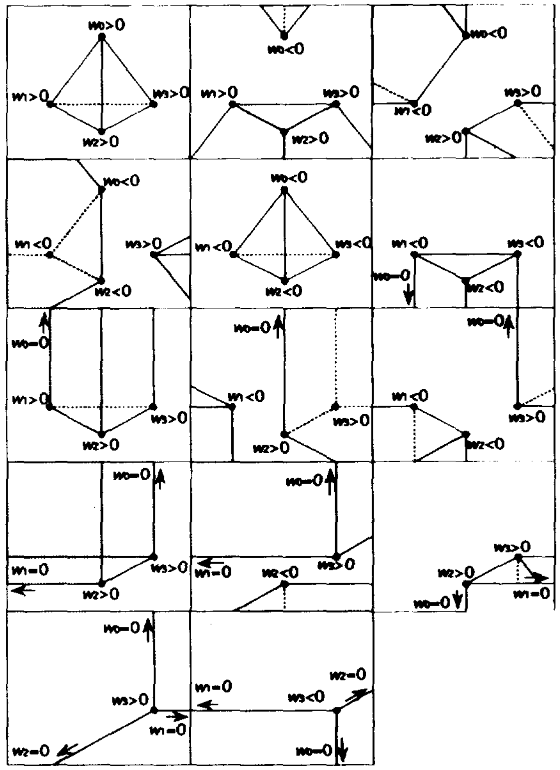


Fig. 5. Examples of homogeneous tetrahedra.

When the weights of the four homogeneous coordinate vectors have the same sign (positive or negative), the set of points expressed by this equation represents an ordinary tetrahedron. This set of points will be called the *Homogeneous Tetrahedron* $\mathbf{V}_0\mathbf{V}_1\mathbf{V}_2\mathbf{V}_3$. Figure 5 shows homogeneous tetrahedra with differently signed weights. There are no homogeneous tetrahedra with four zero weights because vectors with zero weights represent coplanar points (i.e., they lie on the plane at infinity and form a *degenerate* homogeneous tetrahedron). The projective transformation of a homogeneous tetrahedron produces another homogeneous tetrahedron. We may obtain a homogeneous tetrahedron from an ordinary tetrahedron by performing a projective transformation on it. The four faces of a homogeneous tetrahedron are homogeneous triangles, and the six sides are homogeneous line segments.

3. DEFINITIONS OF DUAL FIGURES

3.1 The Principle of Duality

A plane is represented as a four-component nonzero column vector $\gamma = [a\ b\ c\ d]^T$. This vector is called the homogeneous coordinate vector of the

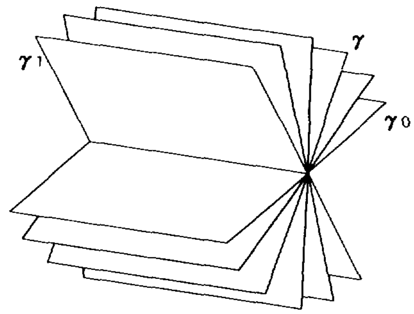


Fig. 6. A dual homogeneous line segment.

plane. Any nonzero multiple of this vector, $\lambda\gamma = [\lambda a\ \ \lambda b\ \ \lambda c\ \ \lambda d]^T$ $(\lambda \neq 0)$ represents the same plane. When a point lies on a plane, the following equation holds:

$$[X\ \ Y\ \ Z\ \ w]\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0. \tag{4}$$

The principle of duality between points and planes can be stated as follows. Every definition remains significant, and every theorem remains true, when we interchange the words "point" and "plane" [Arokiasamy 1989; Chazelle et al. 1983; Coxeter 1969]. By applying the principle of duality to the concepts defined in the previous section, we obtain their duals.

3.2 Duals of Homogeneous Line Segments

We defined a homogeneous line segment as a linear combination of homogeneous coordinate vectors representing two distinct points. The dual of a homogeneous line segment is the set of planes represented by a linear combination of vectors representing two distinct planes.

$$\begin{aligned} \gamma = {} & \xi_0\gamma_0 + \xi_1\gamma_1, \\ & \text{where } (\xi_0, \xi_1) \neq (0,0) \\ & \text{and } (\xi_0, \xi_1 \geq 0 \text{ or } \xi_0, \xi_1 \leq 0). \end{aligned} \tag{5}$$

We will call this set of planes the *Dual Homogeneous Line Segment* $\gamma_0\gamma_1$. A dual homogeneous line segment is a linear pencil of planes defined by the two vectors γ_0 and γ_1. The two planes represented by γ_0 and γ_1 divide the pencil of planes formed by these two planes into two parts. A dual homogeneous line segment represents one of these parts. This is illustrated in Figure 6. The projective transformation of a dual homogeneous line segment produces another dual homogeneous line segment.

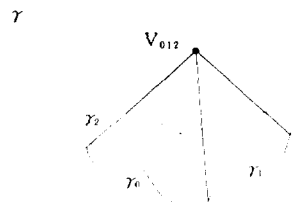


Fig. 7. A dual homogeneous triangle.

3.3 Duals of Homogeneous Triangles

The dual of a homogeneous triangle is defined as a linear combination of vectors representing three distinct planes which have no line in common.

$$\gamma = \xi_0\gamma_0 + \xi_1\gamma_1 + \xi_2\gamma_2, \quad \text{where } (\xi_0, \xi_1, \xi_2) \neq (0,0,0) \quad \text{and } (\xi_0, \xi_1, \xi_2 \geq 0 \text{ or } \xi_0, \xi_1, \xi_2 \leq 0). \tag{6}$$

This set of planes will be called a *Dual Homogeneous Triangle*. A dual homogeneous triangle is a set of planes which all share one common line, as may be seen in Figure 7. The projective transformation of a dual homogeneous triangle produces another dual homogeneous triangle.

3.4 Duals of Homogeneous Polygons

The dual of a homogeneous polygon is a concurrent set of planes. It is defined as a cyclic sequence of the homogeneous coordinate vectors of planes. Since this set of planes is not so intuitive, and is practically impossible to draw, we will not discuss it in detail here. However, the discussions and algorithms concerning homogeneous polygons apply equally to the dual and must not be overlooked.

3.5 Duals of Homogeneous Tetrahedra

Finally, the dual of a homogeneous tetrahedron is a set of planes defined as a linear combination of vectors representing four distinct planes. The four planes must not be concurrent, and each set of three planes must not have a line in common.

$$\gamma = \xi_0\gamma_0 + \xi_1\gamma_1 + \xi_2\gamma_2 + \xi_3\gamma_3, \quad \text{where } (\xi_0, \xi_1, \xi_2, \xi_3) \neq (0,0,0,0) \quad \text{and } (\xi_0, \xi_1, \xi_2, \xi_3 \geq 0 \text{ or } \xi_0, \xi_1, \xi_2, \xi_3 \leq 0). \tag{7}$$

This set of planes will be called the *Dual Homogeneous Tetrahedron* $\gamma_0\gamma_1\gamma_2\gamma_3$.

4. INTERSECTION DETECTION TESTS

4.1 Notation

All intersection detections in this article will be expressed by means of 4×4 determinants. The 4×4 determinant consisting of the homogeneous coordinate vectors of 4 arbitrary points $\mathbf{V}_0, \mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3$ is denoted by S_{0123}. The 4×4 determinant consisting of the homogeneous coordinate vectors of 4 arbitrary planes $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ is denoted by T_{0123}. Thus,

$$S_{0123} = \begin{vmatrix} X_0 & Y_0 & Z_0 & w_0 \\ X_1 & Y_1 & Z_1 & w_1 \\ X_2 & Y_2 & Z_2 & w_2 \\ X_3 & Y_3 & Z_3 & w_3 \end{vmatrix}, \tag{8}$$

and

$$T_{0123} = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \\ d_0 & d_1 & d_2 & d_3 \end{vmatrix}. \tag{9}$$

The two types of determinants are duals.

4.2 Projective Invariance of Intersection Tests

We take four arbitrary homogeneous coordinate vectors $\mathbf{V}_0, \mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3$ representing points, and an arbitrary 4×4 projective transformation matrix A. Here, we denote the determinant consisting of the homogeneous coordinate vectors $\mathbf{V}_0, \mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3$, of four points by $\det(\mathbf{V}_0, \mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3)$. The determinant of the matrix A will also be denoted by $\det(A)$. $\mathbf{V}_0 \cdot A$ denotes the vector $\mathbf{V}_0$ transformed by the matrix A. The following relationship holds between determinants consisting of point vectors before and after a projective transformation:

$$\det(\mathbf{V}_0 \cdot A, \mathbf{V}_1 \cdot A, \mathbf{V}_2 \cdot A, \mathbf{V}_3 \cdot A) = \det(\mathbf{V}_0, \mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3) \cdot \det(A).$$

The projective transformation results in the multiplication of the determinant consisting of the four points by $\det(A)$. Since A is a projective transformation matrix, $\det(A)$ is either positive or negative. If $\det(A)$ is positive, the determinant $\det(\mathbf{V}_0, \mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3)$ does not change its sign after the projective transformation. If $\det(A)$ is negative, the determinant $\det(\mathbf{V}_0 \cdot A, \mathbf{V}_1 \cdot A, \mathbf{V}_2 \cdot A, \mathbf{V}_3 \cdot A)$ changes its sign. The same can be stated for determinants consisting of homogeneous coordinates of planes.

An example of a projective transformation with a matrix having negative determinants is a transformation which changes the orientation of the coordinate system (right handed or left handed). This projective transformation causes the reversal of the signs of all determinants.

The above observation yields the following result. If an intersection test is expressed entirely by means of the sign tests of 4×4 determinants as shown in (8) and (9), the intersection test is invariant under projective transformation if and only if the test gives the same result even when the sign of every determinant is reversed.

4.3 Intersection Tests

We shall now derive the intersection detection tests between the homogeneous primitives and the dual primitives we have defined. These are necessary and sufficient conditions of intersection derived directly from the definitions of the figures. The intersection tests are expressed entirely by means of sign tests of 4×4 determinants. From the result shown in the previous section, the intersection tests are invariant under projective transformation.

Test 4.3.1 *The Containment Test of a Point* $\mathbf{V}_a$ *in a Homogeneous Tetrahedron* $\mathbf{V}_0\mathbf{V}_1\mathbf{V}_2\mathbf{V}_3$. The condition for the point represented by the vector $\mathbf{V}_a$ to be contained in the homogeneous tetrahedron is

$$(S_{a123}, S_{0a23}, S_{01a3}, S_{012a} \geq 0) \vee (S_{a123}, S_{0a23}, S_{01a3}, S_{012a} \leq 0).$$

Test 4.3.1′ *The Containment Test of a Plane* γ_a *in a dual Homogeneous Tetrahedron* $\gamma_0\gamma_1\gamma_2\gamma_3$. The condition for the plane represented by the vector γ_a to be contained in a dual homogeneous tetrahedron is

$$(T_{a123}, T_{0a23}, T_{01a3}, T_{012a} \geq 0) \vee (T_{a123}, T_{0a23}, T_{01a3}, T_{012a} \leq 0).$$

Test 4.3.2 *The Containment Test of a Point* $\mathbf{V}_a$ *in a Homogeneous Triangle* $\mathbf{V}_0\mathbf{V}_1\mathbf{V}_2$. Let $\mathbf{V}_N$ be a vector which forms a homogeneous tetrahedron with the point vectors $\mathbf{V}_0$, $\mathbf{V}_1$, and $\mathbf{V}_2$. The condition for the point represented by $\mathbf{V}_a$ to be contained in the homogeneous triangle is

$$(S_{012a} = 0) \wedge ((S_{a12N}, S_{0a2N}, S_{01aN} \geq 0) \vee (S_{a12N}, S_{0a2N}, S_{01aN} \leq 0)).$$

Test 4.3.2′ *The Containment Test of a Plane* γ_a *in a Dual Homogeneous Triangle* $\gamma_0\gamma_1\gamma_2$. Let γ_N be a vector which forms a dual homogeneous tetrahedron with the plane vectors γ_0, γ_1, and γ_2. The condition for the plane represented by γ_a to be contained in the dual homogeneous triangle is

$$(T_{012a} = 0) \wedge ((T_{a12N}, T_{0a2N}, T_{01aN} \geq 0) \vee (T_{a12N}, T_{0a2N}, T_{01aN} \leq 0)).$$

Test 4.3.3 *The Containment Test of a Point* $\mathbf{V}_a$ *in a Homogeneous Line Segment* $\mathbf{V}_0\mathbf{V}_1$. Let $\mathbf{V}_N$ and $\mathbf{V}_M$ be vectors which form a homogeneous tetrahedron with the vectors $\mathbf{V}_0$ and $\mathbf{V}_1$. The condition for the point represented by $\mathbf{V}_a$ to be contained in the homogeneous line segment is

$$(S_{01Ma}, S_{01aN} = 0) \wedge ((S_{a1MN}, S_{0aMN} \geq 0) \vee (S_{a1MN}, S_{0aMN} \leq 0)).$$

Test 4.3.3′ *The Containment Test of a Plane* γ_a *in a Dual Homogeneous Line Segment* $\gamma_0\gamma_1$. Let γ_N and γ_M be vectors which form a dual homogeneous tetrahedron with the vectors γ_0 and γ_1. The condition for the plane represented by γ_a to be contained in the dual homogeneous line segment is

$$(T_{01Ma}, T_{01aN} = 0) \wedge ((T_{a1MN}, T_{0aMN} \geq 0) \vee (T_{a1MN}, T_{0aMN} \leq 0)).$$

Test 4.3.4 *The Intersection Test of a Homogeneous Line Segment* $\mathbf{V}_a\mathbf{V}_b$ *with a Homogeneous Triangle* $\mathbf{V}_0\mathbf{V}_1\mathbf{V}_2$. The condition for the homogeneous line segment to intersect the homogeneous triangle at a single point is

$$\begin{aligned}&((S_{12ab}, S_{20ab}, S_{01ab} \geq 0) \vee (S_{12ab}, S_{20ab}, S_{01ab} \leq 0))\\ &\quad \wedge((S_{012b} \geq 0 \wedge S_{012a} \leq 0) \vee (S_{012b} \leq 0 \wedge S_{012a} \geq 0))\\ &\quad \wedge((S_{12ab}, S_{20ab}, S_{01ab}) \neq (0,0,0) \wedge (S_{012a}, S_{012b}) \neq (0,0)).\end{aligned}$$

When an intersection point exists, it may be computed as

$$\begin{aligned}&\text{if } S_{012b} - S_{012a} > 0, \quad \mathbf{V} = S_{012b}\mathbf{V}_a - S_{012a}\mathbf{V}_b,\\ &\text{if } S_{012b} - S_{012a} < 0, \quad \mathbf{V} = -S_{012b}\mathbf{V}_a + S_{012a}\mathbf{V}_b,\end{aligned}$$

or,

$$\begin{aligned}&\text{if } S_{12ab} + S_{20ab} + S_{01ab} > 0, \quad \mathbf{V} = S_{12ab}\mathbf{V}_0 + S_{20ab}\mathbf{V}_1 + S_{01ab}\mathbf{V}_2,\\ &\text{if } S_{12ab} + S_{20ab} + S_{01ab} < 0, \quad \mathbf{V} = -S_{12ab}\mathbf{V}_0 - S_{20ab}\mathbf{V}_1 - S_{01ab}\mathbf{V}_2.\end{aligned}$$

Test 4.3.4′ *The Intersection Test of a Dual Homogeneous Line Segment* $\gamma_a\gamma_b$ *with a Dual Homogeneous Triangle* $\gamma_0\gamma_1\gamma_2$. The condition for the dual homogeneous line segment to have a single plane in common with the dual homogeneous triangle is

$$\begin{aligned}&((T_{12ab}, T_{20ab}, T_{01ab} \geq 0) \vee (T_{12ab}, T_{20ab}, T_{01ab} \leq 0))\\ &\quad \wedge((T_{012b} \geq 0 \wedge T_{012a} \leq 0) \vee (T_{012b} \leq 0 \wedge T_{012a} \geq 0))\\ &\quad \wedge((T_{12ab}, T_{20ab}, T_{01ab}) \neq (0,0,0) \wedge (T_{012a}, T_{012b}) \neq (0,0)).\end{aligned}$$

When an intersection plane exists, it may be computed as

$$\begin{aligned}&\text{if } T_{012b} - T_{012a} > 0, \quad \gamma = T_{012b}\gamma_a - T_{012a}\gamma_b,\\ &\text{if } T_{012b} - T_{012a} < 0, \quad \gamma = -T_{012b}\gamma_a + T_{012a}\gamma_b,\end{aligned}$$

or

$$\begin{aligned}&\text{if } T_{12ab} + T_{20ab} + T_{01ab} > 0, \quad \gamma = T_{12ab}\gamma_0 + T_{20ab}\gamma_1 + T_{01ab}\gamma_2,\\ &\text{if } T_{12ab} + T_{20ab} + T_{01ab} > 0, \quad \gamma = -T_{12ab}\gamma_0 - T_{20ab}\gamma_1 - T_{01ab}\gamma_2.\end{aligned}$$

Test 4.4.5 *The Intersection Test of Two Homogeneous Line Segments* $\mathbf{V}_a\mathbf{V}_b$ *and* $\mathbf{V}_0\mathbf{V}_1$. Let $\mathbf{V}_N$ be a vector representing a point not coplanar with the two homogeneous line segments. The condition for the homogeneous line segments to intersect at a single point is

$$\begin{aligned}&(S_{01ab} = 0) \wedge ((S_{1Nab} \geq 0 \wedge S_{0Nab} \leq 0) \vee (S_{1Nab} \leq 0 \wedge S_{0Nab} \geq 0))\\ &\quad \wedge((S_{01Nb} \geq 0 \wedge S_{01Na} \leq 0) \wedge (S_{01Nb} \leq 0 \wedge S_{01Na} \geq 0))\\ &\quad \wedge((S_{1Nab}, S_{0Nab}) \neq (0,0) \wedge (S_{01Na}, S_{01Nb}) \neq (0,0)).\end{aligned}$$

When an intersection point exists, it may be computed as,

$$\begin{aligned}&\text{if } S_{1Nab} - S_{0Nab} > 0, \quad \mathbf{V} = S_{1Nab}\mathbf{V}_0 - S_{0Nab}\mathbf{V}_1,\\ &\text{if } S_{1Nab} - S_{0Nab} < 0, \quad \mathbf{V} = -S_{1Nab}\mathbf{V}_0 + S_{0Nab}\mathbf{V}_1,\end{aligned}$$

or

$$\begin{aligned} &\text{if } S_{01Nb} - S_{01Na} > 0, \qquad \mathbf{V} = S_{01Nb}\mathbf{V}_a - S_{01Na}\mathbf{V}_b, \\ &\text{if } S_{01Nb} - S_{01Na} < 0, \qquad \mathbf{V} = -S_{01Nb}\mathbf{V}_a + S_{01Na}\mathbf{V}_b. \end{aligned}$$

Test 4.3.5′ *The Intersection Test of Two Dual Homogeneous Line Segments* $\gamma_a\gamma_b$ *and* $\gamma_0\gamma_1$. Let γ_N be a vector representing a plane that is not concurrent with the two dual homogeneous line segments. The condition for the dual homogeneous line segments to have a single plane in common is

$$\begin{aligned} (T_{01ab} = 0) &\wedge ((T_{1Nab} \geq 0 \wedge T_{0Nab} \leq 0) \vee (T_{1Nab} \leq 0 \wedge T_{0Nab} \geq 0)) \\ &\wedge ((T_{01Nb} \geq 0 \wedge T_{01Na} \leq 0) \vee (T_{01Nb} \leq 0 \wedge T_{01Na} \geq 0)) \\ &\wedge ((T_{1Nab}, T_{0Nab}) \neq (0,0) \wedge (T_{01Na}, T_{01Nb}) \neq (0,0)). \end{aligned}$$

When an intersection plane exists, it may be computed as,

$$\begin{aligned} &\text{if } T_{1Nab} - T_{0Nab} > 0, \qquad \gamma = T_{1Nab}\gamma_0 - T_{0Nab}\gamma_1, \\ &\text{if } T_{1Nab} - T_{0Nab} < 0, \qquad \gamma = -T_{1Nab}\gamma_0 + T_{0Nab}\gamma_1, \end{aligned}$$

or

$$\begin{aligned} &\text{if } T_{01Nb} - T_{01Na} > 0, \qquad \gamma = T_{01Nb}\gamma_a - T_{01Na}\gamma_b, \\ &\text{if } T_{01Nb} - T_{01Na} < 0, \qquad \gamma = -T_{01Nb}\gamma_a + T_{01Na}\gamma_b. \end{aligned}$$

Test 4.3.6 *The Containment Test of a Point* $\mathbf{V}_a$ *in a Homogeneous Polygon* $\mathbf{V}_0\mathbf{V}_1 \cdots \mathbf{V}_{m-1}$. Let $\mathbf{V}_N$ be a vector representing a point not coplanar with the homogeneous polygon. The containment test for the point in the homogeneous polygon is conducted as follows. The algorithm is shown using a C-like programming language. This algorithm is a variation of the method described in Yamaguchi and Niizeki [1990] and Guibas et al. [1983].

4.4 Duality of Intersection Tests

Note that the intersection tests in the preceding section are presented in pairs. The intersection tests of primitives which are duals are identical in form and can be proved in exactly the same manner. Thus, the tests are duals and may be conducted using the same program or hardware.

4.5 The Selection of Auxiliary Points and Planes

Some of the intersection tests described above used auxiliary points and planes represented by homogeneous coordinate vectors denoted by $\mathbf{V}_N$, $\mathbf{V}_M$, γ_N, and γ_M. These vectors may always be selected from one of the following fundamental vectors.

$$\text{Points: } [1 \;\; 0 \;\; 0 \;\; 0], [0 \;\; 1 \;\; 0 \;\; 0], [0 \;\; 0 \;\; 1 \;\; 0], [0 \;\; 0 \;\; 0 \;\; 1].$$

$$\text{Planes: } [1 \;\; 0 \;\; 0 \;\; 0]^T, [0 \;\; 1 \;\; 0 \;\; 0]^T, [0 \;\; 0 \;\; 1 \;\; 0]^T, [0 \;\; 0 \;\; 0 \;\; 1]^T.$$

```
#define TRUE   -1
#define FALSE  0

index0 = index1 = TRUE;
for (i = 0; i < m; i++) {
        j = (i + 1) % m;
        k = (j + 1) % m;
        if (S_ijkN < 0) {
                if ((S_aijN <= 0)&&(S_ajkN > 0))
                        index0 = !index0;
                if ((S_aijN < 0)&&(S_ajkN >= 0))
                        index1 = !index1;
        } else if (S_ijkN > 0) {
                if ((S_aijN > 0)&&(S_ajkN <= 0))
                        index0 = !index0;
                if ((S_aijN >= 0)&&(S_ajkN < 0))
                        index1 = !index1;
        }
}
if ((index0 != index1)||(index0 != FALSE)) {
        V_a is inside of homogeneous polygon;
} else {
        Va is outside of homogeneous polygon;
}
```

Fig. 8. Containment test in a homogeneous polygon.

5. ADDITIONAL TESTS

The intersection detection tests which were not mentioned above are obtained by combining the above tests. For example, the intersection test of a homogeneous line segment and a homogeneous tetrahedron is the combination of the containment tests of the end points of the homogeneous line segment with respect to the homogeneous tetrahedron and the intersection test between the homogeneous line segment and the four faces of the homogeneous tetrahedron. Degenerate cases are also treated by combining the above tests.

We now examine some additional concepts which are useful in the modeling process.

5.1 Computation of Points and Planes

The intersection tests above were between two sets of points or between two sets of planes. Since three point vectors determine a plane vector, and three plane vectors determine a point vector, we can define a set of planes using point vectors, and a set of points using plane vectors. The following equation computes the plane vector determined by three point vectors and the point vector determined by three plane vectors.

The Vector Representing the Plane γ Determined by 3 Point Vectors $\mathbf{V}_0$, $\mathbf{V}_1$, $\mathbf{V}_2$.

$$\gamma = [\,N_{012_X} \quad N_{012_Y} \quad N_{012_Z} \quad D_{012}\,]^T,$$

$$N_{012_X} = \begin{vmatrix} Y_0 & Z_0 & w_0 \\ Y_1 & Z_1 & w_1 \\ Y_2 & Z_2 & w_2 \end{vmatrix}, \qquad N_{012_Y} = \begin{vmatrix} Z_0 & X_0 & w_0 \\ Z_1 & X_1 & w_1 \\ Z_2 & X_2 & w_2 \end{vmatrix},$$

$$N_{012_Z} = \begin{vmatrix} X_0 & Y_0 & w_0 \\ X_1 & Y_1 & w_1 \\ X_2 & Y_2 & w_2 \end{vmatrix}, \qquad D_{012} = -\begin{vmatrix} X_0 & Y_0 & Z_0 \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}. \tag{10}$$

The Vector Representing the Point $\mathbf{V}$ Determined by 3 Point Vectors γ_0, γ_1, γ_2.

$$\mathbf{V} = [\,X_{012} \quad Y_{012} \quad Z_{012} \quad w_{012}\,],$$

$$X_{012} = \begin{vmatrix} b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \\ d_0 & d_1 & d_2 \end{vmatrix}, \qquad Y_{012} = \begin{vmatrix} c_0 & c_1 & c_2 \\ a_0 & a_1 & a_2 \\ d_0 & d_1 & d_2 \end{vmatrix},$$

$$Z_{012} = \begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ d_0 & d_1 & d_2 \end{vmatrix}, \qquad w_{012} = -\begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{vmatrix}. \tag{11}$$

By converting points into planes, and planes into points, we can apply the above intersection tests to sets of planes defined by point vectors and sets of points defined by plane vectors, respectively. For example, the point-in-polygon test for a polygon defined by plane vectors uses the same algorithm as a polygon defined by point vectors. In these cases, there are more efficient methods of computation for conducting these tests without computing the coordinate vectors of the points or planes directly.

5.2 Convexity of Vertices of Homogeneous Polygons

The convexity of vertices of homogeneous polygons is defined similarly as with ordinary polygons (Figure 9). A vertex is said to be convex when two arbitrary points in the neighborhood of the vertex can be connected by a homogeneous line segment contained completely in the neighborhood. A vertex which is not convex is concave. The definition of the convexity for vertices of ordinary polygons is contained in this definition.

We will derive a convexity test for the vertices of a homogeneous polygon. The homogeneous triangles obtained in an oriented triangulation as seen in Figure 10(b) have the same orientation, when the weights are taken into account. We take an arbitrary point vector $\mathbf{V}_N$, which represents a point not on the plane of the homogeneous polygon. Three vectors of the vertices of a homogeneous triangle obtained by the triangulation, $\mathbf{V}_a$, $\mathbf{V}_b$, $\mathbf{V}_c$, and the point

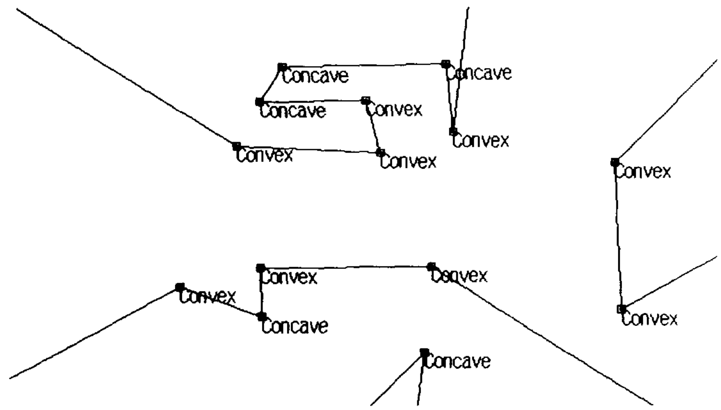


Fig. 9. Convexity of vertices of homogeneous polygons.

vector $\mathbf{V}_N$ form the determinant S_{Nabc}. This determinant S_{Nabc} takes the same sign over all the homogeneous triangles of the homogeneous polygon.

When $\mathbf{V}_N$ is taken so that every S_{Nabc} is positive, and $\mathbf{V}_i$, $\mathbf{V}_j$, and $\mathbf{V}_k$ are consecutive vectors representing the homogeneous polygon, the convexity test of a vertex represented by the vector $\mathbf{V}_j$ of the homogeneous polygon is as follows:

$$\text{if } S_{Nijk} > 0, \quad \mathbf{V}_j \text{ is convex,}$$
$$\text{if } S_{Nijk} < 0, \quad \mathbf{V}_j \text{ is concave.}$$

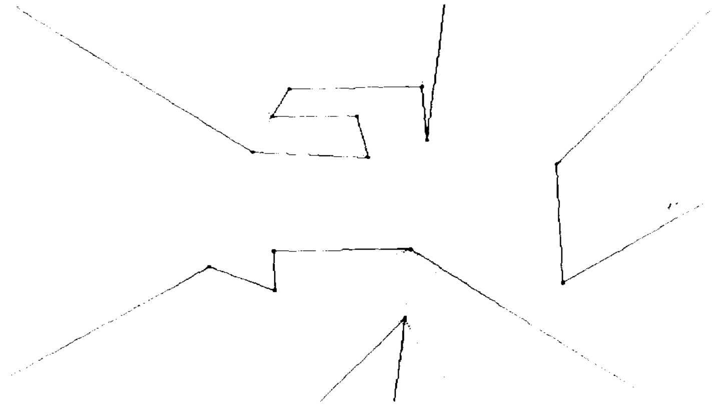
This test is identical to the convexity test for ordinary polygons.

6. APPLICATIONS

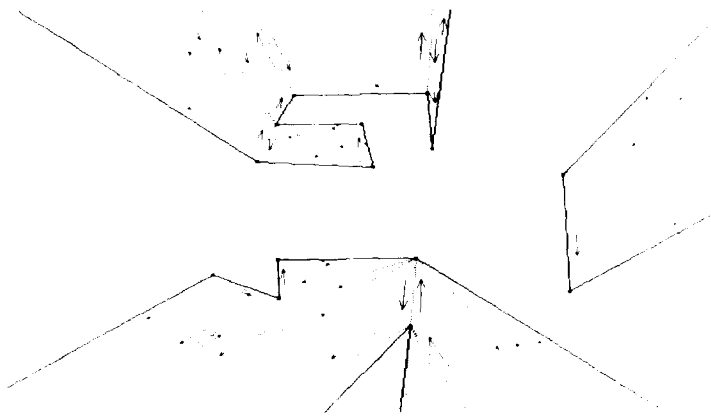
6.1 Triangulation of Homogeneous Polygons

As stated in the previous section, homogeneous polygons can be triangulated into homogeneous triangles in a consistent orientation as shown in Figure 10(b). Here we will present a simple algorithm for the triangulation of an arbitrary homogeneous polygon.

We choose an arbitrary point vector $\mathbf{V}_N$, representing a point not on the plane of the homogeneous polygon. Next, we find three consecutive vertice vectors $\mathbf{V}_i$, $\mathbf{V}_j$, $\mathbf{V}_k$ from the homogeneous polygon whose determinant S_{Nijk} is positive. (If triangulation does not succeed using this sign, we retry the whole process using the other sign.) If no other vertices of the homogeneous polygon are contained in the homogeneous triangle $\mathbf{V}_i\mathbf{V}_j\mathbf{V}_k$, this homogeneous triangle can be cut off from the homogeneous polygon. If this homogeneous triangle cannot be cut off, we try again using another set of vertices. By cutting off a homogeneous triangle, we obtain a homogeneous polygon with one less vertex vector. We continue this process on this new homogeneous polygon until the



(a)



(b)

Fig. 10. Triangulation of homogeneous polygons.

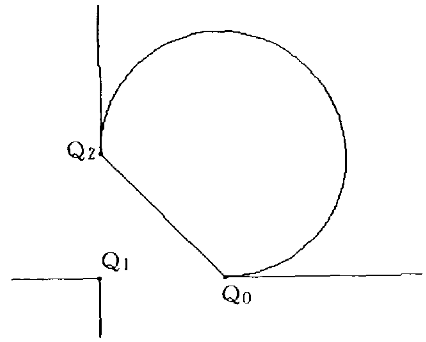
homogeneous polygon has only three vertices left. Then we have succeeded in the triangulation.

This algorithm may also be used to triangulate ordinary polygons. Since the algorithm described here focuses on simplicity, this algorithm is not very efficient. There are much more efficient methods of triangulating a homogeneous polygon, but those methods are beyond the scope of this article.

6.2 Rational Parametric Curves and Surfaces

Rational parametric curve segments and surface patches in the Bézier or B-spline form are now important methods of the representation of geometric forms. Two useful properties of these representations are the convex-hull property and subdivision algorithms. The control points of these curve segments and surface patches are represented as homogeneous coordinate vectors. These curves and surfaces lie entirely in the convex hull of their con-

Fig. 11. A rational parametric Bézier curve segment.



trol point vectors in the four-dimensional homogeneous coordinate space. When the weight of every control point vector has the same sign (positive or negative), the convex-hull property also holds in three-dimensional space. However, when the control point vectors have an arbitrary sign (positive, zero, or negative), the convex-hull property does not hold in three-dimensional space in general.

This makes these representations inconvenient compared to their nonrational counterpart. In the case of the nonrational curves and surfaces, in which the convex-hull property holds always, the convex-hull property is frequently combined with the subdivision algorithm to be used as intersection detection algorithms. Usually, rational curves and surfaces limit the control points to having positive weights when these algorithms are employed. However, the new concepts introduced in this article provide a method of implementing intersection tests for these curves and surfaces using the convex-hull and subdivision properties. We will take rational Bézier curve segments as an example here. As can be seen in Figure 11, the rational Bézier curve segment does not always lie within the triangle formed by the control points. However, if we consider the *homogeneous triangle* formed by the control point vectors, the rational Bézier curve segment can be seen to lie entirely within it.

Higher-degree curve segments and surface patches also lie completely within the bounding homogeneous figures formed by their control point vectors (Figure 12). Thus we may generalize this concept. An intersection detection algorithm may be conducted as follows. We check if any of the homogeneous tetrahedra, homogeneous triangles, or homogeneous line segments formed by the control point vectors of the curve segment or surface patch intersect the figure in consideration. If there is no intersection, the curve segment or surface patch does not intersect the figure. If there is an intersection, we subdivide the curve segment or surface patch and repeat the check. If there is an intersection, and we have reached a specified tolerance, then we may assume that the curve segment or surface patch intersects the object figure.

We must note that there are cases where the control points are in a position where the bounding figure of the curve segment or surface patch

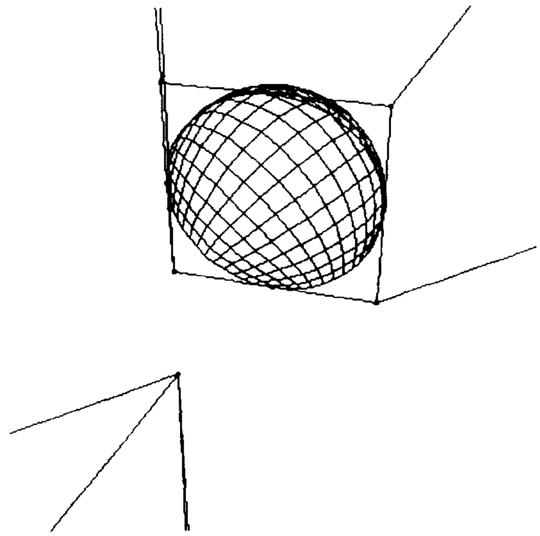


Fig. 12. A rational parametric Bézier surface patch.

covers the whole space, an entire plane, or line. In such cases, we subdivide the curve or surface until the bounding figure reduces to the processable homogeneous primitives we have already described. A more complete description of the processing of parametric curves and surfaces can be found in Yamaguchi and Niizeki [1993; 1994] and Yamaguchi et al. [1991].

We should also note that it is convenient to display control polygons and polyhedral control nets on a screen using homogeneous line segments, since they represent the shape of curve segments and surface patches better than when ordinary line segments are used, especially when the control point vectors have mixed signs. For example, the homogeneous line segments in Figure 11 lie much intuitively closer to the curve segment at the end points when compared to the ordinary triangle formed by the control points. Homogeneous line segments can be displayed using the homogeneous coordinate-clipping technique described in Blinn and Newell [1978].

6.3 Display Algorithms

Hidden-line and surface removal algorithms which conduct intersection tests after a perspective transformation must have some method of handling the homogeneous primitives described in this article. Points positioned behind the view point will have negative weights when the perspective matrix is multiplied. The methods described here provide the theoretic basis for the hidden-line removal algorithms such as those of Watkins [1970] and Warnock [1967], which otherwise would not always operate properly [Yamaguchi and Niizeki 1993].

7. CONCLUSION

We have redefined the basic primitives required in geometric modeling using homogeneous coordinate vectors. We have described the methods of intersection detections between these primitives. These methods are invariant under projective transformation. We have also defined dual primitives which are sets of planes, and described their intersection detection methods. We now have a complete system of geometric intersection tests for primitives defined in homogeneous coordinates. These methods enable us to perform intersection detection tests after an arbitrary projective transformation. We can now detect intersections between primitives with homogeneous coordinate vectors which have weights with arbitrary signs. We have also obtained a dual method of intersection detections for sets of planes.

We have seen that a single geometric processing package or hardware processor can be applied to intersection detection with respect to Euclidean as well as homogeneous and dual primitives. This geometric hardware processor is based solely on the computation of a 4×4 determinant. The intersection tests presented here are also very simple and may be implemented on hardware without much difficulty. We have presented several applications which require these new concepts to process in a uniform manner.

Since our main purpose was to discuss the theoretical foundations of this geometric intersection-testing method, and to find a simple computation basis for geometric hardware we have deliberately left discussions concerning implementation issues such as algorithmic efficiency and numerical accuracy for another article. But the concepts which were described are supersets of the usual concepts we are accustomed to. Hence, the virtues of the 4×4 determinant method are inherent in this method [Yamaguchi 1987]. For example, we may economize computation by repeated use of intermediate computation results of determinants. In this article, we assumed infinite-precision arithmetic because an error-free computation method using variable-length integer arithmetic has been implemented and is being used based on the 4×4 determinant method [Yamaguchi et al. 1993]. There are also efficient intersection tests of curve segments using a recursive determinant computation method [Yamaguchi et al. 1991]. We believe that the identification of these simple and sound geometric primitive concepts and operations will lead to a further understanding of the problems we face in modeling applications today.

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